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HEAVISIDE'S ELECTRIC CIRCUIT THEORY

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FOREWORD

It gives me considerable pleasure to write a foreword to this book by a colleague who has given me much assistance, as we have together made a mathematical attack on some rather difficult engineering problems.

Progressive development in electrical engineering has been rapid, partly because many of the problems encountered have been such as to yield to mathematical analysis. This has made accurate prediction of the behaviour of electric circuits possible. As electrical engineering develops, circuits become more complicated and the growing importance of transient effects makes problems harder to solve. It is evident, therefore, that the electrical engineer must learn to use more powerful mathematical tools. One of the most useful for the analysis of electric circuits is the Heaviside Operational Calculus. This is, in effect, a simple shorthand technique for evaluating the results of Fourier integral analysis.

This Monograph was prepared at the suggestion of Professor Willis Jackson, D.Sc., D.Phil., M.I.E.E., of Manchester University, with the needs of electrical engineering students in mind. It is based on an 'Out-of-Hours' course of lectures delivered by Mr. Josephs to engineers at the Post Office Research Station. The scheme adopted in the original lectures and closely followed in this book is to base electric circuit theory on a far-reaching theorem which, so far, appears to have escaped the notice of engineers. This theorem has been re-constructed from the scattered papers of Heaviside and is probably the last he ever

deduced; for this reason the author has called it 'Heaviside's Last Theorem'. The famous Carson integral equation may be obtained as a corollary to this theorem. It also leads to many others involving Fourier and Bessel integrals, elliptic functions, &c., which have a wider field of application than electric circuits.

It is hoped that this little book will be of service in preparing the ground for a greater use of Heaviside's Operational Calculus and that it will lead to a fuller and more just appreciation of the great work done by Heaviside in connexion with the foundations of electric circuit theory.

W. G. RADLEY

POST OFFICE ENGINEERING
RESEARCH STATION

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CHAPTER I

THE FOUNDATIONS OF ELECTRIC CIRCUIT THEORY

INTRODUCTION

THE object of making an analysis of an electric circuit problem, stated in a most general way, is to formulate as a function of the time the current which flows in any part of an electric network in response to a suddenly applied voltage of arbitrary form. This means that the analysis is not limited to a discussion of steady-state oscillations only, but includes the study of transient phenomena following abrupt circuit changes. Whilst a knowledge of the symbolic steady-state analysis, involving the operator j, is sufficient for the solution of many important technical problems, it is quite inadequate for the solution of many problems which now face the electrical engineer. In telegraphy, for example, it is well known that the signalling speed is always limited by the transient effects; while in telephony, the transient effects form the limiting factor to the distance over which telephonic communication can be established. Consequently the analysis and design of electric networks with special reference to their behaviour in the transient state and when subjected to overlapping transient forces are of considerable importance. There can be little doubt that the best method of attacking the more difficult problems of electric circuit theory is the direct operational method invented by the late Mr. Oliver Heaviside. In this book an endeavour has been made to review and discuss in simple language the application of Heaviside's operational calculus to electric circuit problems.

CIRCUIT PARAMETERS

When tackling a circuit problem for the first time, Heaviside began by writing down the descriptive differ-

ential equations of the problem, which he could readily obtain from known physical laws. The next step lay in the solution of these equations, and it was here that Heaviside found the ordinary classic methods of solving differential equations presented formidable mathematical difficulties and in certain important cases broke down completely. This was the reason why he parted with academic methods and invented mathematical procedure of a revolutionary kind.

We begin by discussing the mathematical basis of circuit theory. A circuit may be defined as a physical system in which the varying quantities may be expressed in terms of time and some other dimension. The characteristics of such a system may be expressed in terms of circuit parameters. In the electric circuit there are three classes of parameters with which we have to deal. These parameters may be defined as follows:

(1) *Definition of the Dissipative Parameter*

The dissipative parameter of an electric circuit is called the resistance. Resistance is defined by the equation, $e = Ri$, which states that the voltage e across a pure resistance is proportional to the current i through it; the proportionality factor being the value of the resistance R. If e is in volts and i in amperes, the resistance R will be in ohms.

(2) *Definition of the Inertia Parameter*

The inertia parameter of an electric circuit is called the inductance. Inductance is defined by the equation, $e = L\frac{di}{dt}$, which states that the voltage e across a pure inductance is proportional to the rate of change of current through it; the proportionality factor being the value of the inductance L. Thus a coil has unit inductance if unit rate of change of current through it produces a unit voltage

across it. If e is in volts, i in amperes and t in seconds, then the inductance L will be in henrys.

(3) *Definition of the Spring Parameter*

The spring parameter of an electric circuit is called the capacitance. Capacitance is defined by the equation, $i = C\frac{de}{dt}$, which states that the current i through a pure condenser is proportional to the rate of change of voltage across it; the proportionality factor being the value of the capacity C. Thus a condenser has unit capacity if unit rate of change of voltage across it produces a unit current through it. The above definition may be written in the form

$$e = \frac{1}{C}\int i\,dt = \frac{q}{C}$$

which states that the capacity C is given by the charge q required to produce unit voltage across it. If e is in volts and the charge q in coulombs, the capacity C will be in farads.*

In order to fix ideas, consider a simple electric circuit consisting of a resistance R, an inductance L and a capacity C, in series. Let i denote the current in the circuit and e the applied voltage. Then the potential drop across the resistance is iR, the inductance drop is $L\frac{di}{dt}$, and the drop across the condenser is $\frac{q}{C}$, where q is the charge on the condenser.

Now from Kirchhoff's law relating to the potential drop around the circuit, we may write

$$iR + L\frac{di}{dt} + \frac{q}{C} = e$$

* For a discussion of dimensions, see *Physical Constants* by W. H. J. Childs (Methuen's Monographs on Physical Subjects).

multiplying both sides of this equation by i, we get

$$i^2R+\frac{d}{dt}\cdot\frac{Li^2}{2}+\frac{d}{dt}\cdot\frac{q^2}{2C}=ei$$

since $i=dq/dt$. The first term i^2R expresses the rate at which electrical energy is being converted into heat. The second term $\frac{d}{dt}\cdot\frac{Li^2}{2}$ expresses the rate of increase of the magnetic energy. The third term $\frac{d}{dt}\cdot\frac{q^2}{2C}$ expresses the rate of increase of the electric energy. The right-hand side is clearly the rate at which the impressed force is delivering energy to the circuit. Thus we may regard a resistance element as a device for converting electrical energy into heat. An inductance element may be regarded as a device for storing energy in the magnetic field. A capacitance element may be regarded as a device for storing energy in the electric field.

KIRCHHOFF'S LAWS

There are two distinct problems involved in circuit analysis. The first is the formulation of the descriptive differential equations from known physical laws, and the second is the mathematical solution of these equations. The descriptive equations of a circuit problem may be established in a number of different ways. For example, they may be based on Maxwell's dynamical theory. There can be little doubt, however, that the simplest basis for the equations of electric circuit theory are to be found in Kirchhoff's Laws. For the electric circuit these laws state that:

(1) In any closed circuit in a network of conductors the algebraic sum of the potential differences is zero.

(2) The algebraic sum of the currents which meet at any point in a network of conductors is zero.

The law which states that the sum of all the voltages about a closed circuit is zero follows from the principle

of the conservation of energy. All voltages must be included, not only the applied voltages, and the voltages due to the circuit elements, but any voltages induced by coupling with neighbouring circuits. The law which states that the sum of the currents flowing into a junction point is zero follows from the principle of the conservation of electricity.

As an example of the appropriate mode of setting up the circuit equations, consider the two-mesh network shown in Fig. 1.

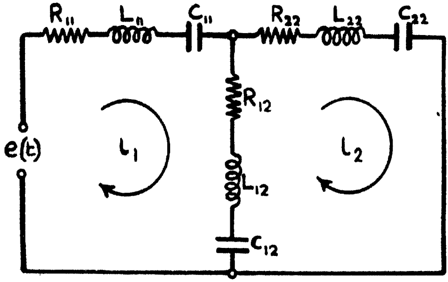

FIG. 1.—Two-mesh network.

Let $e(t)$ represent the applied voltage as a function of the time t. Let the two-mesh currents be represented by i_1 and i_2 respectively. Then, if the positive direction of the current in each mesh is taken to be clockwise, the resulting current in the mutual branch (consisting of a resistance R_{12}, an inductance L_{12} and a capacity C_{12}) can be written down from Kirchhoff's second law. Applying Kirchhoff's first law to the two meshes, we obtain

$$e(t) = \left\{(R_{11} + R_{12})i_1 + (L_{11} + L_{12})\frac{di_1}{dt} + \left(\frac{1}{C_{11}} + \frac{1}{C_{12}}\right)\int i_1\,dt\right\}$$
$$- \left\{R_{12}i_2 + L_{12}\frac{di_2}{dt} + \frac{1}{C_{12}}\int i_2\,dt\right\}$$

$$0 = \left\{(R_{22} + R_{12})i_2 + (L_{22} + L_{12})\frac{di_2}{dt} + \left(\frac{1}{C_{22}} + \frac{1}{C_{12}}\right)\int i_2\,dt\right\}$$
$$- \left\{R_{12}i_1 + L_{12}\frac{di_1}{dt} + \frac{1}{C_{12}}\int i_1\,dt\right\}$$

If we let p denote the operator $\frac{d}{dt}$ and p^{-1} the operation $\int dt$, the above equations can be written in the Ohm's law form,

$$e(t) = Z_{11}(p)i_1 - Z_{12}(p)i_2$$
$$0 = -Z_{12}(p)i_1 + Z_{22}(p)i_2$$

where the impedance functions are

$$Z_{11}(p) = (R_{11} + R_{12}) + p(L_{11} + L_{12}) + p^{-1}\left(\frac{1}{C_{11}} + \frac{1}{C_{12}}\right)$$

$$Z_{22}(p) = (R_{22} + R_{12}) + p(L_{22} + L_{12}) + p^{-1}\left(\frac{1}{C_{22}} + \frac{1}{C_{12}}\right)$$

$$Z_{12}(p) = R_{12} + pL_{12} + p^{-1}\left(\frac{1}{C_{12}}\right)$$

It will be observed that our definition of a circuit includes mechanical systems not usually regarded as circuits. On account of the similarity of their descriptive differential equations, however, they may be treated as such. All rigid mechanical oscillating systems of a single degree of freedom may be treated as mechanical circuits, together with coupled combinations of such systems. This includes such systems as oscillograph vibrators, telephone diaphragms, rotating shafts and deflecting beams, &c.

THE CANONICAL EQUATIONS

We proceed now to a discussion of the canonical equations of electric circuit theory. Our reasons for doing this are to show how the ordinary method of solving sets of simultaneous differential equations breaks down in the general case; and to show how Heaviside overcame the

formidable mathematical difficulties associated with classic analysis by attacking the problem from a new viewpoint.

An electric network is a connected set of circuits or meshes each of which can be regarded as made up of resistance, inductance and capacitance elements. Suppose the network has n independent meshes numbered from 1 to n, and suppose the corresponding mesh currents be denoted by i_1, i_2 . . . up to i_n. Let R_{gg}, L_{gg}, C_{gg} represent the total resistance, inductance and capacity in series in mesh g, and let R_{gh}, L_{gh}, C_{gh} represent the mutual resistance, inductance and capacity between the circuits g and h. Kirchhoff's first law enables us to write the circuit equations that must hold for each and every one of the n meshes. Thus, if $e(t)$ represents the voltage applied to the terminals of the first mesh, we may write

$$e(t) = R_{11}i_1 + L_{11}\frac{di_1}{dt} + \frac{1}{C_{11}}\int i_1\,dt = Z_{11}(p)i_1$$

but this circuit is coupled to a second circuit in all three ways, so we may write

$$e(t) = Z_{11}(p)i_1 + \left(R_{12} + pL_{12} + \frac{1}{pC_{12}}\right)i_2$$

$$\therefore\ e(t) = Z_{11}(p)i_1 + Z_{12}(p)i_2$$

If there are n coupled circuits in the network, there will be n such terms, and we may write

$$e(t) = Z_{11}(p)i_1 + Z_{12}(p)i_2 + \ldots + Z_{1(n-1)}(p)i_{n-1} + Z_{1n}(p)i_n$$

If $Z_{22}(p)$ represents the impedance function of the second mesh, then we may write for the second mesh

$$0 = Z_{21}(p)i_1 + Z_{22}(p)i_2 + \ldots + Z_{2(n-1)}(p)i_{n-1} + Z_{2n}(p)i_n$$

A similar equation may be written down for the other meshes of the network. Hence the complete equations for the general case may be written in the form

$$\left.\begin{aligned} e(t) &= Z_{11}(p)i_1 + \ldots + Z_{1n}(p)i_n \\ &\cdots\cdots\cdots \\ 0 &= Z_{n1}(p)i_1 + \ldots + Z_{nn}(p)i_n \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

These equations, together with a description of the initial condition of the network, completely specify its performance. Their solution and interpretation form the subject of electric circuit theory and in the study of their solution we shall find the most direct introduction to Heaviside's operational calculus. It should be noted that the canonical equations (1) are capable of immediate generalization. The variable to be determined need not be the current. It may equally well be any other linear variable with which we may happen to be concerned. It must be noted that the circuit parameters are fixed and that $R_{gh} = R_{hg}$, $L_{gh} = L_{hg}$ and $C_{gh} = C_{hg}$. This reciprocal relationship means that there are no concealed sources or sinks of energy in the network. To solve the set of equations (1) means that we must deduce mathematical expressions for the currents which substituted into the right-hand members will give the left-hand members. These expressions must also satisfy the terminal conditions at reference time $t = 0$.

THE STEADY-STATE SOLUTION

In taking up the mathematical solution of the canonical equations (1) we shall start with the exponential solution. This is of fundamental importance to the engineer because it serves as the basis of the steady-state solution of alternating current problems. Thus we consider exponential currents and voltages of the form $\exp(j\omega t)$, where $j\omega$ has its usual significance in alternating current theory. If we assume that the applied voltage $e(t) = e^{j\omega t}$, we know that the expression for the resultant current in each branch of the network will contain an exponential steady-state term of the same form. This follows because an exponential does not change in form when differentiated or integrated. Consequently the set of differential equations (1) may be reduced to algebraic equations because $\frac{d}{dt}$ (or p) may be replaced by $j\omega$ and $\int dt$ (or p^{-1}) replaced by $(j\omega)^{-1}$.

Having 'algebrized' the differential equations (1) it is now a simple matter to determine any steady-state variable I_K by the elementary methods applicable to first-degree algebraic equations. Hence

$$I_K = \frac{M_{1K}(j\omega)}{D(j\omega)} \qquad (K = 1, 2, \ldots n)$$

where $M_{1K}(j\omega)$ is the minor determinant of the first row and Kth column, and

$$D(j\omega) = \begin{vmatrix} Z_{11}(j\omega) & \ldots & Z_{1n}(j\omega) \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ Z_{n1}(j\omega) & \ldots & Z_{nn}(j\omega) \end{vmatrix}$$

is the determinant of the coefficients.

It is an easy matter to put I_K in an Ohm's law form, thus:

$$I_K = 1/Z(j\omega)$$

where the impedance function $Z(j\omega)$ is the ratio of the two determinants $D(j\omega)/M_{1K}(j\omega)$.

Now assume that the applied voltage varies as $E \cos \omega t$. Writing $E \cos \omega t$ in its exponential form we have

$$E \cos \omega t = \frac{E}{2}(e^{j\omega t} + e^{-j\omega t})$$

Consequently the applied voltage may be considered in two parts, each of which can be applied separately, and the results added, because we are dealing with a linear network with fixed parameters. Using the Ohm's law form for I_K we have

$$I_K = \frac{Ee^{j\omega t}}{2Z(j\omega)} + \frac{Ee^{-j\omega t}}{2Z(-j\omega)}$$

In general, $Z(j\omega)$ and $Z(-j\omega)$ are conjugate complex numbers, and consequently

$$Z(\pm j\omega) = |Z(j\omega)|\, e^{\pm j\phi}$$

Hence we obtain

$$I_K = \frac{E}{2\,|Z(j\omega)|}(e^{j\omega t}.e^{-j\phi} + e^{-j\omega t}.e^{j\phi})$$

which can be written as

$$I_K = \frac{E}{|Z(j\omega)|} \cos(\omega t - \phi)$$

This in concentrated form contains the whole theory of the symbolic solution of alternating current problems. It will be observed that only steady-state currents are involved and such currents are 'forced' currents. They vary with the time in precisely the same manner as do the electromotive forces. Such currents are, however, only part of the total currents; the complete solution must contain transient as well as steady-states. In mathematical language the complete solution of the canonical equations (1) must include both 'particular' and 'complementary' solutions.

If it is found that the steady-state solution of a problem is sufficient, then the need for the use of the Heaviside calculus does not arise. In cases such as this the well-known symbolic method outlined above is sufficient. Perhaps it is now generally recognized that it is not possible to predict completely the performance of an electric network from a single steady-state solution. The growing practical importance of non-periodic phenomena demands a knowledge not only of the steady-state but also of the transient response. Both of these result simultaneously from an operational solution.

THE TRANSIENT SOLUTION

In order to obtain the transient terms we must find the complementary function which satisfies the canonical equations (1). Let us assume that a solution of these equations exists in the form

$$i_1 = F_{11}e^{\lambda t},\ i_2 = F_{22}e^{\lambda t},\ \ldots\ \&c.$$

The correctness of this assumption may be checked by substituting into the set of equations (1) and putting $e(t)$ equal to zero. If we make these substitutions and

then divide out the exponential the following set of algebraic equations is obtained:

$$\left.\begin{array}{l} 0 = Z_{11}(\lambda)F_{11} + \ldots + Z_{1n}(\lambda)F_{1n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 0 = Z_{n1}(\lambda)F_{n1} + \ldots + Z_{nn}(\lambda)F_{nn} \end{array}\right\}$$

Such a set of algebraic equations has a solution only when the determinant of the system vanishes; that is, when $D(\lambda) = 0$. The equation $D(\lambda) = 0$ is known as the determinantal equation of the network; and if $\lambda_1, \lambda_2, \lambda_3 \ldots$ &c., are the roots, then transient expressions of the form

$$i_g = F_{g1}e^{\lambda_1 t} + F_{g2}e^{\lambda_2 t} + \ldots \text{ \&c.}$$

will satisfy the canonical equations (1). This is the form of the transient portion of the solution. The values of $F_{g1}, F_{g2}, F_{g3} \ldots$ &c., are constants of integration which must satisfy imposed boundary conditions. This introduces us to the idea of terminal conditions which are of fundamental importance in circuit analysis. The terminal conditions denote the state of the system when the e.m.f. is applied. The number of known terminal conditions required to satisfy the complementary function must be equal to the number of roots of the equation $D(\lambda) = 0$. If this number of independent terminal conditions is not known the complete solution of the canonical equations (1) by this method is indeterminate.

BREAKDOWN OF THE CLASSIC METHOD

So far we have followed the ordinary classic method of solving the canonical equations (1). The practical difficulties of this method of solution are:

(1) The determination of the arbitrary constants F_{g1}, F_{g2}, F_{g3}, ... &c., from known terminal conditions; and

(2) The determination of the roots $\lambda_1, \lambda_2, \lambda_3 \ldots$ &c., from the determinantal equation $D(\lambda) = 0$.

With regard to (1): in the general case it is quite impossible to determine terminal conditions from the know-

ledge that if an e.m.f. be applied to a dissipative network at time $t = 0$, then at that instant, all the currents in the inductances and all the charges on the condensers are zero. With regard to (2): in the general case it is quite impossible to determine the complex roots of the equation $D(\lambda) = 0$. It was at this point that Heaviside parted with classic analysis and invented his operational calculus, with which we will now be directly concerned.

THE HEAVISIDE METHOD

From the preceding analysis it can be seen that Heaviside was concerned with the determination of the dependent variables $i_1, i_2, \ldots i_n$, of the canonical equations (1), as functions of the time t, with the terminal condition that these variables are all zero for the independent variable $t < 0$. If the number of closed circuits or meshes is made infinite and the size of the circuit elements infinitesimal, the canonical equations (1) change into a partial differential equation of the wave-motion type. In his attack on the general problem Heaviside made use of the principle of superposition. When several electromotive forces are applied simultaneously to a network of fixed parameters, each produces its own effect independently of the others. Thus the effect of each alone may be calculated and the results added to obtain a resultant due to the simultaneous action of the various electromotive forces. This suggested to Heaviside that he could deduce a superposition theorem expressing the independent variables in terms of voltage and admittance functions which have been made to satisfy the necessary terminal conditions of the network. Thus it became apparent that these variables are capable of expression in the form $\sum\limits_{0}^{t} e(t-\psi)\delta\{h(\psi)\}$, provided the auxiliary variables h are dependent upon the general equations (1) and are all identically zero for $t < 0$.

Heaviside found that if he replaced $e(t)$ in the equations (1) by a very simple discontinuous function, namely, a unit

voltage applied suddenly at reference time $t = 0$, he could obtain for the prescribed boundary conditions an auxiliary set of equations in the auxiliary variables $h_1, h_2, h_3, \ldots h_n$. Thus

$$\left.\begin{aligned} [1] &= Z_{11}(p)h_1 + \ldots + Z_{1n}(p)h_n \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ 0 &= Z_{n1}(p)h_1 + \ldots + Z_{nn}(p)h_n \end{aligned}\right\} \quad \ldots \quad (2)$$

where [1], the function on the left-hand side, written in accordance with the Heaviside notation as unity, is identically zero for $t < 0$ and unity for $t \geqslant 0$ and $h_1, h_2, \ldots h_n$ are identically zero for $t < 0$. Thus the auxiliary equations (2) are made to satisfy the necessary terminal condition of the canonical equations (1), and consequently from the principle of superposition it is possible to show that solutions for $i_1, i_2, i_3, \ldots$ &c., exist in the form

$$i(t) = \frac{d}{dt}\int_0^t e(t - \psi)h(\psi)\,d\psi \quad \ldots \quad (3)$$

where the symbol ψ represents the variable of integration and $i(t)$ represents the current which will flow when the impressed voltage is given by the time function $e(t)$. A short method of establishing equation (3) is given in Chapter 8. The function $h(t)$ is called the 'Indicial Admittance' and represents the current which flows in the mesh considered when a unit voltage [1] is applied to the input terminals of the network. It can be seen that the superposition theorem expressed by equation (3) mathematically relates the current in any mesh to the type of applied voltage and to the parameters and connexions of the system. Thus Heaviside showed that the solution of the canonical equations (1) depends entirely on the set of auxiliary equations (2) and the superposition theorem (3)

THE OPERATIONAL EQUATION

The important problem of determining $h = h(t)$ remains, and Heaviside's method of attack was to reduce the

differential equations (1) formally to algebraic equations by replacing the differential operator d/dt by the symbol p and the operation $\int dt$ by p^{-1}. He also put the applied voltage $e(t)$ equal to [1] and thus obtained auxiliary equations from which he could readily obtain a symbolic solution for a particular variable. This symbolic solution he called the 'operational equation' of the problem. From the foregoing it can be seen that the operational equation of a problem is the full equivalent of the descriptive differential equations, and must, therefore, contain the information necessary to the solution provided the significance of the symbolic operator p can be correctly determined. The operational equation of a problem obtained by algebraic processes from equations (2) may be written formally as

$$h_K(t)[1] = \frac{1}{Z_K(p)}[1] \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$(K = 1, 2, 3, \ldots n)$$

where $h(t)$, the indicial admittance, represents the variable to be determined and $Z(p)$ represents a generating impedance function containing the p operators.

The unit function [1] represents a discontinuous function of the time which is zero until t equals zero and unity thereafter. This unit function is impor ant because it is the simplest possible discontinuous function and is always tacitly assumed in electrical theory. As an example of this fact, suppose it is required to find the current response i of a circuit containing a resistance R and an inductance L in series, to which an alternating voltage $E \cos \omega t$ is applied at reference time $t = 0$. The differential equation is

$$L\frac{di}{dt} + iR = E \cos \omega t$$

for which we obtain by the ordinary method

$$i = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t + \phi) - \frac{ERe^{-\frac{Rt}{L}}}{R^2 + \omega^2 L^2}$$

where $$\phi = \tan^{-1}\frac{R}{\omega L}$$

Now suppose that $E = 100$ volts, $R = 1$ ohm, $L = 0{\cdot}01$ henry and $\omega = 100\pi$, then the expression for i shows that 0·002 seconds after the switch is closed the current is 17·063 amps. The expression also shows that 0·002 seconds before the switch is closed (i.e. at the time of – 0·002 seconds) the current is – 20·912 amps. This absurd result is avoided by making an assumption that the time t must be positive. Yet there is nothing in the mathematics from the setting up of the differential equation to its solution to justify this assumption.

The fact is that the differential equation has not been set

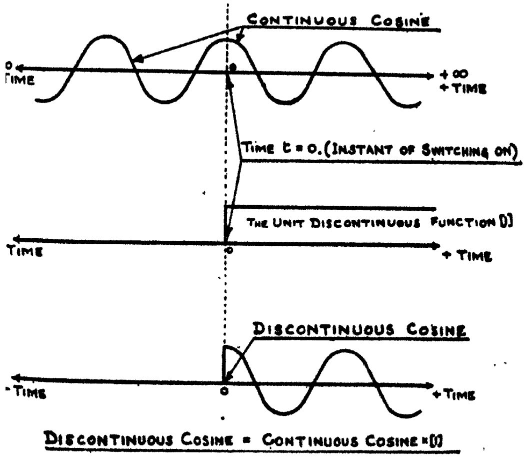

FIG. 2.—Continuous and discontinuous cosines.

up correctly; the applied voltage is not $100 \cos \omega t$ but $100 \cos \omega t[1]$, where $[1]$ is the unit discontinuous function; the effect of operating on $100 \cos \omega t$ by $[1]$ is illustrated in Fig. 2. If the differential equation is set up correctly the factor $[1]$ will then appear in the solution for i, thus showing mathematically that i is zero until t is positive. However, all trouble is avoided by assuming that t is always positive, and this simple example is sufficient to show that the existence of a unit discontinuous function $[1]$ is always tacitly assumed in alternating current theory although not always explicitly stated. Since the unit function $[1]$ occurs on both sides of the operational equation (4) its presence need not be explicitly stated and in general it will be omitted from operational equations.

CHAPTER 2

THE EXPANSION THEOREM

AN EXPERIMENTAL DERIVATION

IN order to interpret his symbolic or operational equation (4), Heaviside at first proceeded in an experimental manner. He directly compared the operational equations for certain specific problems with explicit solutions obtained by classic methods. Thus he was led to assign a definite meaning to his operators and he obtained by induction rules for solving the operational equations for those problems in which the classic methods failed him. A little later, however, he discovered certain relationships, which enabled him to proceed deductively as well as inductively. One such relationship was his famous Expansion Theorem. Heaviside established this theorem in a number of different ways, one of which will be reviewed here.

Consider the operator $1/Z_K(p)$ of equation (4); from the set of auxiliary equations (2), we may write

$$\frac{1}{Z_K(p)} = \frac{M_{1K}(p)}{D(p)}$$

where $M_{1K}(p)$, the minor of the first row and Kth column, is of lower order than $D(p)$, the determinant of the auxiliary equations (2).

Let $\lambda_1, \lambda_2, \ldots \lambda_n$, denote the n roots of $D(p) = 0$.

$$\therefore D(p) = (p - \lambda_1)(p - \lambda_2) \ldots (p - \lambda_n).$$

Now assume that in general,

$$\frac{M(p)}{D(p)} = \frac{A_1}{p - \lambda_1} + \frac{A_2}{p - \lambda_2} + \ldots + \frac{A_n}{p - \lambda_n}$$

where $A_1, A_2, \ldots A_n$, are constants and independent of p.

$$\begin{aligned}\therefore\ M(p) = {} & A_1(p-\lambda_2)(p-\lambda_3)\ldots(p-\lambda_n) + \\ & A_2(p-\lambda_1)(p-\lambda_3)\ldots(p-\lambda_n) + \\ & \vdots \\ & A_n(p-\lambda_1)(p-\lambda_2)\ldots(p-\lambda_{n-1}).\end{aligned}$$

Since both sides of this equation are identical for all values of p we may substitute λ_1 for p throughout; this gives

$$M(\lambda_1) = A_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_n)$$

Differentiating $D(p)$ with respect to p, we get

$$\begin{aligned}\frac{\partial D(p)}{\partial p} = {} & (p-\lambda_2)(p-\lambda_3)\ldots(p-\lambda_n) + \\ & (p-\lambda_1)(p-\lambda_3)\ldots(p-\lambda_n) + \\ & \vdots \\ & (p-\lambda_1)(p-\lambda_2)\ldots(p-\lambda_{n-1})\end{aligned}$$

Hence $\quad D'(p) = (\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_n)$

Thus we obtain

$$\begin{aligned}& M(\lambda_1) = A_1 D'(\lambda_1) \\ & \therefore\ A_1 = M(\lambda_1)/D'(\lambda_1)\end{aligned}$$

In like manner we obtain

$$\begin{aligned}& A_2 = M(\lambda_2)/D'(\lambda_2) \\ & \vdots \\ & A_n = M(\lambda_n)/D'(\lambda_n)\end{aligned}$$

Thus, when all the roots are unequal, we have

$$\frac{M(p)}{D(p)} = \frac{M(\lambda_1)}{D'(\lambda_1)}\cdot\frac{1}{p-\lambda_1} + \frac{M(\lambda_2)}{D'(\lambda_2)}\cdot\frac{1}{p-\lambda_2} + \ldots + \frac{M(\lambda_n)}{D'(\lambda_n)}\cdot\frac{1}{p-\lambda_n}$$

So that

$$\frac{1}{Z(p)}=\sum_{K=1}^{K=n}\frac{M(\lambda_K)}{(p-\lambda_K)D'(\lambda_K)}$$

which is an explicit operational expression for the indicial admittance function $h_K(t)$. This expansion can be expressed in a more convenient form. For put, as in Chapter 1,

$$Z(\lambda)=D(\lambda)/M(\lambda)$$

So

$$\frac{dZ(\lambda)}{d\lambda}=\frac{M(\lambda)D'(\lambda)-D(\lambda)M'(\lambda)}{[M(\lambda)]^2}.$$

Now

$$D(\lambda_K)=0$$
$$Z'(\lambda_K)=D'(\lambda_K)/M(\lambda_K)$$

Thus we may write

$$h_K(t)=\sum_{K=1}^{K=n}\frac{1}{(p-\lambda_K)Z'(\lambda_K)}$$

It will be observed that each term of the above expansion yields a linear differential equation which may be written as

$$u_r=\frac{1}{Z'(\lambda_r)}\cdot\frac{1}{(p-\lambda_r)}\quad(r=1,2,3\ldots n)$$

$$=\frac{1}{Z'(\lambda_r)}\cdot\frac{1}{p}\cdot\frac{1}{\left(1-\frac{\lambda_r}{p}\right)}$$

Expanding the last term on the right-hand side of this equation by the binomial theorem, we get

$$u_r=\frac{1}{Z'(\lambda_r)}\cdot\frac{1}{p}\cdot\left\{1+\frac{\lambda_r}{p}+\left(\frac{\lambda_r}{p}\right)^2+\left(\frac{\lambda_r}{p}\right)^3+\ldots\right\}$$

$$=\frac{1}{\lambda_r Z'(\lambda_r)}\left\{\frac{\lambda_r}{p}+\left(\frac{\lambda_r}{p}\right)^2+\left(\frac{\lambda_r}{p}\right)^3+\ldots\right\}$$

The question now arises: What significance is to be given to the operator p^{-n} acting on the unit operand [1]?

Assuming that the usual scheme of integration holds for this operator, we have by definition

$$p^{-1} = \int dt$$

$$\therefore\ p^{-1}.p^{-1} = \frac{t^2}{2}$$

$$\therefore\ p^{-1}.p^{-2} = \frac{t^3}{2.3}$$

from which it can be seen that

$$p^{-n} = \frac{t^n}{n!}$$

when n is some real number. Following the natural suggestion that we substitute $t^n/n!$, for the operator p^{-n}, in the operational expansion for u_r, we get

$$u_r = \frac{1}{\lambda_r Z'(\lambda_r)} \cdot \left\{ \lambda_r t + \frac{\lambda_r^2 t^2}{2!} + \frac{\lambda_r^3 t^3}{3!} + \ldots + \frac{\lambda_r^n t^n}{n!} + \ldots \right\}$$

It can be seen at once that this is equivalent to

$$u_r = \frac{1}{\lambda_r Z'(\lambda_r)} \{ e^{\lambda_r t} - 1 \}$$

Substituting in the expansion for $h_K(t)$ we obtain

$$h_K(t) = \sum_{K=1}^{K=n} \frac{e^{\lambda_K t} - 1}{\lambda_K Z'(\lambda_K)}$$

$$= \sum_{K=1}^{K=n} \frac{e^{\lambda_K t}}{\lambda_K Z'(\lambda_K)} - \sum_{K=1}^{K=n} \frac{1}{\lambda_K Z'(\lambda_K)}$$

Looking back at the operational expansion for $1/Z(p)$, it can be seen that the last term of the above equation is $1/Z(0)$ by putting $p = 0$; so that

$$h_K(t) = \frac{1}{Z(0)} + \sum_{K=1}^{K=n} \frac{e^{\lambda_K t}}{\lambda_K Z'(\lambda_K)} \quad . \quad . \quad . \quad (5)$$

It will be noted that equation (5) is an explicit expression

for $h_K(t)$, which we arrived at experimentally by using an unestablished operational process, and illustrates the type of inductive analysis originally used by Heaviside. The summation indicates that we are to write as many terms as there are roots of the equation $Z(p) = 0$, and into each insert the value of one of these roots. In equation (5) it can be seen that the first term is obviously the steady-state current, and the summation is of exactly the form of the complementary function for the differential equations of the network, where the constants of integration, the cause of the breakdown of the classic method, are now replaced by explicit expressions in terms of the generating impedance function. Equation (5) is the famous Heaviside Expansion Theorem.

TRANSIENT OSCILLATIONS IN A SERIES CIRCUIT

As an example of the application of Heaviside's expansion theorem to an electric circuit problem, suppose it is required to find expressions for the current i in response to a direct voltage E applied suddenly (at reference time $t = 0$) to a circuit consisting of a resistance R, inductance L and a capacity C in series.

From the expansion theorem we have

$$i(t) = \frac{E}{Z(0)} + E\sum_{K=1}^{K=n} \frac{e^{\lambda_K t}}{\lambda_K Z'(\lambda_K)}$$

where $\lambda_1, \lambda_2, \ldots \lambda_n$, are the n roots of the equation $Z(p) = 0$, and

$$Z'(\lambda_K) = \left(\frac{\partial Z(p)}{\partial p}\right)_{p=\lambda_K} \quad (K = 1, 2, 3, \ldots n)$$

The first term $E/Z(0)$ is obviously the steady-state current, and the summation indicates that we are to write as many terms as there are roots of $Z(p) = 0$, and into each insert the value of one of these roots.

The circuit equation for the series circuit is

$$iR + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = E$$

Replacing $\frac{d}{dt}$ by p and $\int dt$ by p^{-1}, we get

$$i(R + pL + 1/pC) = E$$
$$\therefore\ i = E/Z(p)$$

where $$Z(p) = R + pL + \frac{1}{pC}$$

The two roots of the equation $Z(p) = 0$ are

$$\lambda_1 = -\alpha + j\omega$$

and $$\lambda_2 = -\alpha - j\omega$$

where $$\alpha = \frac{R}{2L} \text{ and } \omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$$

Differentiating $Z(\lambda)$ with respect to λ, we get

$$\frac{\partial Z(\lambda)}{\partial \lambda} = L - \frac{1}{\lambda^2 C}$$

Introducing the values of λ_1 and λ_2 we get

$$\left[\frac{\partial Z(\lambda)}{\partial \lambda}\right]_{\lambda=\lambda_1} = L - \frac{1}{C(-\alpha + j\omega)^2} = \frac{2j\omega L}{-\alpha + j\omega}$$

Similarly by replacing ω by $-\omega$, we get

$$\left[\frac{\partial Z(\lambda)}{\partial \lambda}\right]_{\lambda=\lambda_2} = L - \frac{1}{C(-\alpha - j\omega)^2} = \frac{-2j\omega L}{-\alpha - j\omega}$$

Hence $$\lambda_1 Z'(\lambda_1) = 2j\omega L$$

and $$\lambda_2 Z'(\lambda_2) = -2j\omega L$$

For $\lambda = 0$ we have $(1/\lambda C) = \infty$, and consequently $Z(0) = \infty$. Thus the first term $E/Z(0)$ of the expansion theorem becomes E/∞, showing that the steady-state current is zero. Substituting the above results in the expansion theorem, we obtain

$$i = Ee^{-\alpha t}\left\{\frac{e^{j\omega t}}{2j\omega L} - \frac{e^{-j\omega t}}{2j\omega L}\right\}$$

$$\therefore\ i(t) = \frac{E}{\omega L} e^{-\alpha t} \sin \omega t$$

It can be seen that this equation represents a damped oscillation, where the damping factor $\alpha = R/2L$, and the frequency is given by

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

If, however, $\frac{1}{LC} < \frac{R^2}{4L^2}$, we notice that ω now becomes a pure imaginary. Consequently, we put $\omega = j\omega_1$, in the expression for $i(t)$, and we obtain

$$i = \frac{E}{\omega_1 L} e^{-\alpha t} \sin j\omega_1 t$$

but
$$\sin j\omega_1 t = j \sinh \omega_1 t$$

Hence
$$i = \frac{E}{\omega_1 L} e^{-\alpha t} \sinh \omega_1 t$$

Thus it can be seen that when $\frac{1}{LC} < \frac{R^2}{4L^2}$, the current diminishes asymptotically to zero, and the motion is aperiodic instead of oscillatory.

In the critical case, however, when $\frac{1}{LC} = \frac{R^2}{4L^2}$, the two roots λ_1 and λ_2 are equal and the expansion theorem is no longer applicable. We may readily arrive at the expression for the current in this case by noting that when $\lambda_1 \rightarrow \lambda_2$, then $\omega \rightarrow 0$. Consequently, for very small differences between the roots λ_1 and λ_2, ω is very small and $\sin \omega t = \omega t$. Putting $\sin \omega t = \omega t$ in the equation for $i(t)$, we get

$$i = \frac{E}{\omega L} e^{-\alpha t} . \omega t = \frac{Et}{L} e^{-\alpha t}$$

Now this equation is independent of ω and consequently must hold also for the limiting value $\lambda_1 = \lambda_2$ and $\omega = 0$.

The current in this case also approaches zero asymptotically. Collecting results we see that if

$$\frac{1}{LC} > \frac{R^2}{4L^2}, \text{ then } i = \frac{E}{\omega L}e^{-\alpha t}\sin \omega t$$

$$\frac{1}{LC} = \frac{R^2}{4L^2}, \text{ then } i = \frac{Et}{L}e^{-\alpha t}$$

$$\frac{1}{LC} < \frac{R^2}{4L^2}, \text{ then } i = \frac{E}{\omega_1 L}e^{-\alpha t}\sinh \omega_1 t$$

These results are well known. The point that we wish to emphasize is that they were arrived at by a simple algebraic process without having to determine integration constants from known terminal conditions.

PRACTICAL APPLICATION OF THE EXPANSION THEOREM

It has been shown that Heaviside's expansion theorem gives an explicit relationship between the impedance function $Z(p)$ and the indicial admittance function $h(t)$. For electric circuits, where the number of roots involved is small, the expansion theorem enables one to calculate $h(t)$ with very little labour and with the use of simple mathematics. Thus the theorem has proved useful for analysing the transient responses of air-cored transformers and intervalve couplings. It can be applied to all kinds of coupled circuits and, in some cases, to transmission lines. In all these cases the roots of the determinantal equation are easy to locate. When these roots are not easy to locate, that is, when the network has four or more distinct circuits, we run into trouble. This trouble is not due to any fault of the theorem itself, but simply because the engineer encounters practical difficulties when finding the roots of higher-degree algebraic equations, especially when some or all of the roots are complex. Thus great labour is involved in applying the expansion theorem to lumped networks of more than four degrees of freedom.

It must be noted that the partial fraction expansion upon

which the expansion theorem solution depends imposes certain conditions upon the impedance function $Z(p)$. Thus the impedance function must have no zero root and no repeated roots. In all passive electric networks these conditions are satisfied. The case of equal roots, which may occur when the network contains an internal source of energy, such as an amplifier, can be treated by assuming unequal roots and then letting these roots approach equality as a limit. In general, the expansion theorem gives the indicial admittance function (whenever this function is capable of such expression) in terms of the normal or characteristic vibrations of the circuits concerned.

CHAPTER 3

EXTENSION OF THE EXPANSION THEOREM

THE INDICIAL ADMITTANCE FUNCTION

It was stated in Chapter 1 that by virtue of the superposition theorem expressed by equation (3), the problem of circuit analysis was reduced to the determination of the indicial admittance function $h(t)$. In other words, the mathematical formulation of the current flowing in any part of the network in response to a unit voltage (zero before, unity after time $t = 0$) is of fundamental importance. It was shown in Chapter 2 that a formal expression for the indicial admittance function is furnished by the Heaviside expansion theorem, thus

$$h(t) = \frac{1}{Z(0)} + \sum_{K=1}^{n} \frac{e^{\lambda_K t}}{\lambda_K Z'(\lambda_K)} \quad . \quad . \quad . \quad (5)$$

If it is possible to locate the roots of the equation $Z(\lambda) = 0$, the expansion theorem furnishes a practical and working solution. In many important circuit problems, however, the location of the roots of $Z(\lambda) = 0$, is quite impossible, and consequently the expansion theorem as it stands is only of limited practical utility. The way Heaviside overcame this limitation will now be shown.

Putting $t = 0$, in the expansion theorem, we find that

$$h(0) = \frac{1}{Z(0)} + \sum_{K=1}^{n} \frac{1}{\lambda_K Z'(\lambda_K)}$$

which represents the initial conditions in the network, that is, the distribution of current at the instant the switch is closed. For all electric networks met with in practice $h(0)$ is zero, and we may write

$$\frac{1}{Z(0)} = -\sum_{K=1}^{n} \frac{1}{\lambda_K Z'(\lambda_K)}$$

Substituting this value in the expansion theorem, we obtain

$$h(t) = -\sum_{K=1}^{n} \frac{1}{\lambda_K Z'(\lambda_K)} \cdot \{1 - e^{\lambda_K t}\}$$

Expanding the exponential, $h(t)$ becomes

$$h(t) = \frac{t}{1!}\sum \frac{1}{Z'(\lambda_K)} + \frac{t^2}{2!}\sum \frac{\lambda_K}{Z'(\lambda_K)} + \frac{t^3}{3!}\sum \frac{\lambda_K^2}{Z'(\lambda_K)} + \dots$$
$$+ \frac{t^n}{n!}\sum \frac{\lambda_K^{n-1}}{Z'(\lambda_K)} + \dots$$

It is apparent that this series for $h(t)$ may be obtained from the partial fraction expansion of the impedance function $1/Z(p)$, derived in Chapter 2: for we may write this expansion in the form

$$h(t) = \frac{1}{p}\sum_{K=1}^{n} \frac{1}{Z'(\lambda_K)} \left\{1 - \frac{\lambda_K}{p}\right\}^{-1}$$

Expanding the second factor of the summation by the binomial theorem and arranging in inverse powers of p, we get

$$h(t) = \frac{1}{p}\sum \frac{1}{Z'(\lambda_K)} + \frac{1}{p^2}\sum \frac{\lambda_K}{Z'(\lambda_K)} + \frac{1}{p^3}\sum \frac{\lambda^2{}_K}{Z'(\lambda_K)} + \dots$$
$$+ \frac{1}{p^n}\sum \frac{\lambda^{n-1}}{Z'(\lambda_K)} + \dots$$

By comparing this expansion with the expansion in powers of t, we see at once that the coefficient p^{-n} is identical with the coefficient $t^n/n!$. Consequently, it can be seen from these equations that if we expand the function $1/Z(p)$ in the form

$$\frac{1}{Z(p)} = \sum_{K=0}^{\infty} \frac{a_K}{p^K} \quad (K = 0, 1, 2, 3 \dots)$$

then the indicial admittance function is given by

$$h(t) = \sum_{K=0}^{\infty} \frac{a_K t^K}{K!}$$

which is obtained from the operational expansion by replacing p^{-n} by $t^n/n!$.

It will be observed that in making this extension Heaviside discovered a powerful method of evaluating $h(t)$ which does not depend upon the location of the roots of any equation and consequently may be used for those cases to which the expansion theorem cannot be applied readily.

POWER SERIES SOLUTIONS

The power series method outlined above was much used by Heaviside and depended upon the expansion of the function $1/Z(p)$ in inverse powers of p and then replacing p^{-n} by $t^n/n!$. This power series method has the great advantage that it can be readily carried out to obtain a result and involves only elementary mathematics.

As a simple example we will now apply this method to a circuit consisting of an inductance L in series with a resistance R. The differential equation is, in the above notation

$$L\frac{dh}{dt} + Rh = [1]$$

The operational equation is obtained by replacing d/dt by p, and we have

$$h = \frac{1}{R + pL}$$

This equation must now be expanded in inverse powers of p. Thus

$$h = \frac{1}{pL} \cdot \frac{1}{1 + \frac{a}{p}}$$

where $a = R/L$.

Writing this equation in the form

$$h=\frac{1}{R}\cdot\frac{a}{p}\cdot\left(1+\frac{a}{p}\right)^{-1}$$

and expanding in inverse powers of p, we obtain

$$h=\frac{1}{R}\left\{\frac{a}{p}-\left(\frac{a}{p}\right)^2+\left(\frac{a}{p}\right)^3-\left(\frac{a}{p}\right)^4+\ldots\right\}$$

and replacing p^{-n} by $t^n/n!$, we get

$$h=\frac{1}{R}\left\{at-\frac{(at)^2}{2!}+\frac{(at)^3}{3!}-\ldots\right\}$$

which can be recognized as

$$h(t)=\frac{1}{R}\cdot(1-e^{-Rt/L})$$

the well-known indicial admittance function for this circuit. If we had written the operator as $\frac{1}{R}\cdot\frac{a/p}{1+a/p}$, and then replaced p^{-1} by $t/1!$, we should have obtained an incorrect result. In general, an operational equation is converted into an explicit power series solution only by the proper choice of expansion of the reciprocal of the impedance function.

The process of expanding $1/Z(p)$ in such a form as to permit of its being converted into an explicit solution is what Heaviside calls 'algebrizing' the operator. The general process of expansion applicable to all cases where a power series solution of an electric circuit problem exists, may be stated formally as follows:

Put
$$\frac{1}{Z(p)}=\frac{1}{Z(1/x)}=\psi(x)$$

then expanding $\psi(x)$ as a Taylor's series, we get

$$\psi(x)=\psi(0)+x\psi'(0)+\frac{x^2}{2!}\psi''(0)+\ldots+\frac{x^n}{n!}\psi^n(0)+\ldots$$

where
$$\psi^n(0)=\left\{\frac{d^n}{dx^n}\psi(x)\right\}_{x=0}$$

If now we put

$$x^n = \frac{1}{p^n}, \text{ and } \frac{\psi^n(0)}{n!} = A_n$$

we obtain what is called an 'asymptotic' expansion, namely

$$\frac{1}{Z(p)} = A_0 + \frac{A_1}{p} + \frac{A_2}{p^2} + \ldots + \frac{A_n}{p^n} + \ldots$$

It will be recalled that an asymptotic expansion is defined as the expansion of a function in inverse powers of the argument. The theory of asymptotic series occupies an important place in modern analysis. With the question of divergence, convergence and summability to which pure analysis is largely restricted we are not concerned, but only with Heaviside's methods of asymptotic expansion. If the above asymptotic series is regarded as an expansion in the variable p, instead of as a purely symbolic expansion, it will be found to have only a limited range of convergence. This fact may be neglected, however, because the series we are really concerned with is obtained by replacing p^{-n} by $t^n/n!$, namely, the series

$$h(t) = A_0 + \frac{A_1 t}{1!} + \frac{A_2 t^2}{2!} + \ldots + \frac{A_n t^n}{n!} + \ldots$$

which is called the 'associated function' of the former. It can be shown that if the function $1/Z(p)$ satisfies the conditions for an asymptotic expansion, then the resulting power series solution obtained as above is always convergent.

From the foregoing example of the rise of current in a circuit of inductance L and resistance R, it can be seen that the derivation of the explicit power series is a simple process. The process of 'algebrizing' the operators which arise in electric circuit theory usually involves only a straightforward application of the binomial theorem. As an example of 'algebrizing', suppose it is required to obtain an expression for the charge Q, in a series circuit of resistance R, inductance L and capacity C, in response to

the application of a one-volt battery. The differential equation for the charge Q is

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C}=[1]$$

Thus replacing d/dt by the symbol p, we may write

$$Q\left(p^2L+pR+\frac{1}{C}\right)=1, \qquad t>0$$

from which we obtain the operational equation

$$Q=\frac{1}{p^2L+pR+1/C}$$

which may be written as

$$Q=\frac{1}{p^2L}\left(1+\frac{\phi_1}{p}+\frac{\phi_2}{p^2}\right)^{-1}$$

where $\qquad \phi_1=R/L, \qquad \phi_2=1/LC.$

Expanding Q by the binomial theorem, we obtain

$$Q=\frac{1}{p^2L}\left\{1-\left(\frac{\phi_1}{p}+\frac{\phi_2}{p^2}\right)+\left(\frac{\phi_1}{p}+\frac{\phi_2}{p^2}\right)^2-\left(\frac{\phi_1}{p}+\frac{\phi_2}{p^2}\right)^3+\ldots\right\}$$

Arranging this expression in inverse powers of p, we obtain

$$Q=\frac{1}{L}\left\{\frac{1}{p^2}-\frac{a_1}{p^3}-\frac{a_2}{p^4}+\frac{a_3}{p^5}+\ldots\right\}$$

where the a's are readily expressed in terms of ϕ_1 and ϕ_2. The explicit power series solution may now be obtained by replacing p^{-n} by $t^n/n!$. This example shows that the Heaviside process of 'algebrizing' is formally straightforward: more difficult examples are discussed in Chapter 4. It will be observed that in 'algebrizing' the original differential equations no consideration was given to the terminal conditions; it is one of the advantages of the Heaviside method that the terminal conditions in linear networks are automatically accounted for. It is found that a power series solution does not exist for all types of problem relating to transmission lines, although a closely

related series may often be derived. When, however, a power series solution does exist it is easier to derive than that of any other form of solution. A critical review of the scope and value of Heaviside's power series method in electric circuit theory is given in Chapter 4.

THE EXPANSION THEOREM FOR ALTERNATING VOLTAGES

The Heaviside expansion theorem is not limited to the determination of the building-up of the response of a network to a direct voltage, but can be easily extended to include the case of the response to alternating voltage. Suppose that instead of applying to the network the unit discontinuous voltage [1] we apply the unit discontinuous cosine, cos ωt[1]. This discontinuous cosine is illustrated in Fig. 2, and for purposes of analysis may be regarded as the real part of $e^{j\omega t}$[1], where $\omega/2\pi$ represents the frequency of the applied voltage. It is shown in Chapter 8 that if the applied voltage is of the form $Ee^{j\omega t}$[1], instead of the unit function [1], Heaviside's expansion theorem takes the more general form

$$i(t) = \frac{Ee^{j\omega t}}{Z(j\omega)} + E\sum_{K=1}^{K=n} \frac{e^{\lambda_K t}}{(\lambda_K - j\omega)Z'(\lambda_K)}$$

If the applied voltage has some phase angle ψ, i.e. $E \cos (\omega t + \psi)$, then the applied voltage must be written as the real part of the function $Ee^{j\psi}e^{j\omega t}$[1]. Thus the expansion theorem now becomes

$$i(t) = \frac{Ee^{j\psi}e^{j\omega t}}{Z(j\omega)} + Ee^{j\psi}\sum_{K=1}^{K=n} \frac{e^{\lambda_K t}}{(\lambda_K - j\omega)Z'(\lambda_K)}$$

The first term on the right-hand side of this equation represents the steady-state current, whilst the summation terms give the transient oscillations. In applying this theorem to the solution of any problem, it should be noted that only the real parts must be taken, for, in general

$$Z(j\omega) = R + jX$$

and taking the real part of $i(t)$, we get

$$i(t) = \frac{E\cos(\omega t + \psi - \phi)}{\sqrt{R^2 + X^2}} + E\sum_{K=1}^{K=n} \frac{\cos(\psi + \theta) . e^{\lambda_K t}}{\sqrt{\lambda_K{}^2 + \omega^2} Z'(\lambda_K)}$$

where $\quad \phi = \tan^{-1}\dfrac{X}{R}$

and $\quad \theta = \tan^{-1}\dfrac{\omega}{\lambda_K}$

The application of this expression to a simple problem should make the matter clear. Consider a condenser C in series with a resistance R to which we apply an alternating voltage $E \sin \omega t$, the switch being closed at the zero of voltage. Let it be required to determine the current response $i(t)$. It can be seen at once that

$$\psi = -\frac{\pi}{2}$$

and $$Z(\lambda) = R + \frac{1}{\lambda C}$$

Thus $$\lambda_1 = -\frac{1}{CR}$$

Also $$Z'(\lambda) = -\frac{1}{\lambda^2 C}$$

$$\therefore \quad Z'(\lambda_1) = -CR^2$$

Substituting in the equation for $i(t)$, we obtain

$$i(t) = \frac{E\cos\left\{\left(\omega t - \frac{\pi}{2}\right) - \phi\right\}}{\sqrt{R^2 + 1/\omega^2 C^2}} - \frac{Ee^{-\frac{t}{CR}}\cos\left(\frac{\pi}{2} + \theta\right)}{CR^2\sqrt{\omega^2 + 1/C^2R^2}}$$

which reduces to

$$i(t) = E\left\{\frac{\sin(\omega t - \phi)}{\sqrt{R^2 + 1/\omega^2 C^2}} - \frac{e^{-\frac{t}{CR}}}{\omega C(R^2 + 1/\omega^2 C^2)}\right\}$$

the well-known transient response of the condenser circuit to a sinusoidal voltage

It will now be shown that we get exactly the same expression by using the original expansion theorem expressed by equation (5) and then using the $h(t)$ function in the superposition theorem expressed by equation (3). From equation (5) it is a simple matter to show that the indicial admittance function $h(t)$ is given by

$$h(t) = \frac{1}{R}e^{-\frac{t}{CR}}$$

Now $E(t) = E \sin \omega t$ and $E(0) = 0$, and consequently equation (3) may be used in the form

$$i(t) = \int_0^t h(t-\rho)E'(\rho)\,d\rho$$

where $$E'(\rho) = \omega E \cos \omega\rho = \frac{\omega E}{2}(e^{j\omega\rho} + e^{-j\omega\rho})$$

Thus we have

$$i(t) = \frac{\omega E}{2R}\int_0^t e^{-\frac{t-\rho}{CR}}(e^{j\omega\rho} + e^{-j\omega\rho})\,d\rho$$

Performing the indicated integration, we find that $i(t)$ reduces to the well-known expression for the alternating current transient as before.

Limiting our consideration to sinusoidal voltages, it may be remarked that since we can obtain the formal solution of an alternating current transient problem directly from the extended form of Heaviside's expansion theorem, why bother about the definite integration involved in the application of the superposition theorem to the indicial admittance function $h(t)$? If the roots of the equation $Z(\lambda) = 0$ are easily located and are finite in number and different from each other, then the evaluation by the extended form of expansion theorem is preferable. This expansion theorem solution, however, requires the response of an electrical network to be expressed as a sum of char-

acteristic oscillations or pulses, a form of solution which in many practical cases does not admit of easy interpretation or calculation, and in many transmission line problems does not exist. The superposition theorem can always be applied to any class of network problem provided the indicial admittance function $h(t)$ has been evaluated. This theorem can deal with any type of applied voltage and can always be evaluated. If in any particular case the superposition integral is difficult to evaluate formally, it is always possible to evaluate it by graphical or mechanical processes. On these facts rests the practical as distinguished from the purely theoretical value of the superposition formula.

SUBSIDENCE OF THE CURRENT IN A NETWORK

The original Heaviside expansion theorem is not limited to the determination of the building-up of the response of a network to a direct voltage, but also gives correctly the subsidence of this response when the voltage is removed suddenly. Suppose that the voltage has been applied for a sufficiently long period to ensure the wiping out of the transient oscillations, then the current in the circuit is given by the first term of the expansion theorem, namely

$$I_1 = \frac{E}{Z(0)}$$

Suppose that instead of removing or short circuiting the applied voltage, we let it remain and introduce to the network an exactly similar voltage but of opposite sign. It can be seen that the effect will be the same as if the original voltage was suddenly removed. This new voltage will produce a current given by the expression

$$I_2 = -\frac{E}{Z(0)} - E\sum_{K=1}^{K=n} \frac{e^{\lambda_K t}}{\lambda_K Z'(\lambda_K)}$$

Owing to the linear character of the network, the resulting

current will be the sum of the two currents I_1 and I_2. Hence we have

$$I_1 + I_2 = -E \sum_{K=1}^{K=n} \frac{e^{\lambda_K t}}{\lambda_K Z'(\lambda_K)}$$

which is a formula expressing the subsidence of the current in the network when the voltage is suddenly removed.

CHAPTER 4

LADDER NETWORKS

ARTIFICIAL LINES

THE power series method described in Chapter 3 was much used by Heaviside. It has the great advantage that it can be readily carried out to obtain a result, and the corresponding disadvantage that this result (unless the series can be recognized and summed) is often in a form that converges slowly for large values of the time t.

It is an easy matter for an engineer dealing with simple networks to formulate the indicial admittance as a power series by expanding asymptotically the reciprocal of the impedance function. If the network contains transformers or bridge sections, these should first be replaced by their equivalent circuits. In this chapter we propose to show how Heaviside dealt with such complicated structures as artificial lines. Artificial lines are of practical importance in their use for balancing telephone cables and duplexing telegraph cables and other purposes. They also find application in the study of electrical phenomena connected with transmission problems on long lines.

The artificial line is a ladder structure consisting of lumped impedance Z_1 in series with the line and lumped impedances Z_2 in shunt across the line. We regard a ladder network as a periodic structure composed of a series of T sections connected in tandem as shown in Fig. 3.

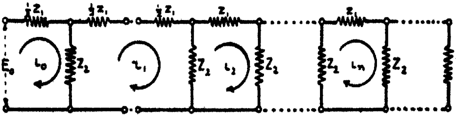

FIG. 3.—Ladder network.

We will assume that the voltage is applied at the middle of the initial or zero'th section as shown in Fig. 3. It will also be observed that the network is terminated by its characteristic impedance Z_0 so that the reflected wave is suppressed.

In order to calculate the transient response of a ladder network of the type shown in Fig. 3 it is first necessary to derive an operational equation which formulates the propagation phenomena. Begin by writing down the voltage drop in the nth section. From Kirchhoff's law we get for the nth section

$$0 = (Z_1 + 2Z_2)i_n - Z_2(i_{n-1} + i_{n+1})$$

where i_n = mesh current in the nth section
i_{n-1} = " " " " $(n-1)$th section
i_{n+1} = " " " " $(n+1)$th "

Since the network is closed by its characteristic impedance Z_0, the reflected wave is suppressed, and consequently we may assume that a solution for i_n exists in the form

$$i_n = ae^{-ny}$$

where a and y are independent of n, the number of sections. Substituting in the Kirchhoff difference equation, we obtain

$$0 = (Z_1 + 2Z_2)ae^{-nv} - Z_2ae^{-nv}(e^v + e^{-v})$$

From this equation it is an easy matter to show that

$$e^{\pm v} = (\sqrt{1+\phi} \pm \sqrt{\phi})^2$$

where $$\phi = \frac{Z_1}{4Z_2}$$

In artificial line theory we are mostly concerned with the entering current i_0 and the current in the nth section. If E_0 is the voltage applied at the mid-series position of the zero'th section we may write

$$i_0 = a = E_0/Z_0$$

and consequently $$i_n = E_0e^{-nv}/Z_0$$

An expression for Z_0 may be readily obtained by considering

the first T section terminated by Z_0; we have, looking into the terminated T section,

$$Z_0 = \tfrac{1}{2}Z_1 + \frac{Z_2(\tfrac{1}{2}Z_1 + Z_0)}{Z_2 + \tfrac{1}{2}Z_1 + Z_0}$$

from which we obtain

$$Z_0 = \sqrt{Z_1 Z_2(1 + \phi)}$$

Consequently the expression for i_n may be written as

$$i_n = \frac{E_0(\sqrt{1+\phi} - \sqrt{\phi})^{2n}}{\sqrt{Z_1 Z_2(1+\phi)}} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which is a convenient form for the expansion of an operational equation and the investigation of transient phenomena in artificial lines.

ARTIFICIAL TELEGRAPH CABLE

We will now apply equation (6) to the determination of the indicial admittances of an artificial telegraph cable of negligible inductance and leakance. Such an artificial cable is shown in Fig. 4, where

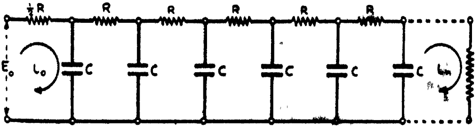

FIG. 4.—Artificial telegraph cable.

$$Z_1 = R, \text{ the series resistance elements}$$

and $\quad Z_2 = \dfrac{1}{pC}$, the shunt capacity elements.

In order to obtain an equation for the input current i_0, we put $n = 0$ in equation (6), thus

$$i_0 = E_0/\sqrt{Z_1 Z_2(1+\phi)}$$

Substituting for Z_1 and Z_2 and putting E_0 equal to [1], we

see that the indicial admittance function $h_0(t)$ can be obtained from the operational equation,

$$h_0 = 1\Bigg/\sqrt{\frac{R}{pC}+\frac{R^2}{4}},$$

For brevity put $y = 4/CR$ and then expand the denominator in a series of inverse powers of p, thus

$$h_0 = \frac{2}{R}\left(1+\frac{y}{p}\right)^{-\frac{1}{2}}$$

$$= \frac{2}{R}\left(1 - \frac{1}{2}\cdot\frac{y}{p} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{y^2}{p^2} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{y^3}{p^3} + \dots\right)$$

Replacing p^{-n} by $t^n/n!$ in the above expansion, we obtain

$$h_0 = \frac{2}{R}\left\{1 - \left(\frac{yt}{2}\right) + \frac{1.3}{(2!)^2}\left(\frac{yt}{2}\right)^2 - \frac{1.3.5}{(3!)^2}\left(\frac{yt}{2}\right)^3 + \dots\right\}$$

where $y = 4/CR$.

For large values of t this series does not converge rapidly enough for easy computation and consequently an effort must be made to recognize and sum the series. At this point we will consider classes of functions which are defined by series expansions, for example

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

and

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

are series definitions of well-known functions. Likewise the Bessel functions $J_n(x)$ and $I_n(x)$ are defined, when n is zero or a positive integer, by the absolutely convergent series:

$$J_n(x) = \frac{x^n}{2^n(n!)}\left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \dots\right\}$$

and

$$I_n(x) = \frac{x^n}{2^n(n!)}\left\{1 + \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} + \dots\right\}$$

For large values of x, the $J_n(x)$ function is oscillatory, and ultimately behaves as $\sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{2n+1}{4}\pi\right)$. The $J_n(x)$ function, however, increases indefinitely with x and behaves ultimately as $e^x/\sqrt{2\pi x}$.

Now Bessel functions have been extensively calculated and fairly complete tables are available. Consequently, expressions containing Bessel functions can often be written down immediately by consulting tables. Thus if we can manage to express a slowly convergent power series solution in terms of Bessel functions, then the arithmetical work is greatly facilitated. The series expansion given for h_0, however, is so slowly convergent for very large values of t that its use for calculation purposes is not justified and consequently we must try to express this series in terms of tabulated functions.

If we put $n = 0$ in the series expansion of $I_n(x)$, and then multiply $I_0(x)$ by e^{-x} term by term, we obtain on combining terms of the same powers of x, the expansion

$$e^{-x}I_0(x) = 1 - x + \frac{1\cdot 3}{(2!)^2}x^2 - \frac{1\cdot 3\cdot 5}{(3!)^2}x^3 + \dots$$

which can be made identical with the series given for h_0 by putting $x = yt/2$. Thus we may write for the power series expansion of h_0, the compact formula,

$$h_0(t) = \frac{2}{R}e^{-\frac{yt}{2}}I_0\left(\frac{yt}{2}\right)$$

which can be rapidly evaluated for all values of t from the tabulated values of the function $e^{-x}I_0(x)$ given in books of mathematical tables.

We will now derive the power series solution for the indicial admittance of the nth section. We begin by writing equation (6) in the form

$$h_n(t) = \frac{(\sqrt{1+\phi} + \sqrt{\phi})^{-2n}}{\sqrt{Z_1 Z_2(1+\phi)}}$$

from which we will develop the operational formula.

Now $\quad (\sqrt{1+\phi}+\sqrt{\phi})^{-2n} = \phi^{-n}(1+\sqrt{1+\phi^{-1}})^{-2n}$

and $\quad \sqrt{Z_1Z_2(1+\phi)} = \sqrt{\phi Z_1Z_2}\sqrt{1+\phi^{-1}}$

Since $\quad \sqrt{\phi Z_1Z_2} = R/2$

We may write

$$h_n = \frac{2}{R}\cdot 2^n\left(\frac{2}{pCR}\right)^n F_1(p)F_2(p)$$

where the factors $F_1(p)$ and $F_2(p)$ are defined as

$$F_1(p) = \frac{1}{\sqrt{1+4/pCR}}$$

and $\quad F_2(p) = (1+\sqrt{1+4/pCR})^{-2n}$

In order to derive the required power series expansion each factor must be expanded by the binomial theorem and arranged in inverse powers of p. After this the two resulting series must be multiplied together and rearranged in powers of $1/p$. Applying this process, we get the series

$$h_n = \frac{2}{2^nR}\left\{\left(\frac{2}{pCR}\right)^n - \frac{(2n+2)}{2(1!)}\left(\frac{2}{pCR}\right)^{n+1} + \frac{(2n+3)(2n+4)}{2^2(2!)}\left(\frac{2}{pCR}\right)^{n+2} - \cdots\right\}$$

Replacing p^{-n} by $t^n/n!$, we get

$$h_n = \frac{2}{2^nR}\left\{\frac{1}{n!}\left(\frac{2t}{CR}\right)^n - \frac{(2n+2)}{2(1!)(n+1)!}\left(\frac{2t}{CR}\right)^{n+1} + \frac{(2n+3)(2n+4)}{2^2(2!)(n+2)!}\left(\frac{2t}{CR}\right)^{n+2} - \cdots\right\}$$

For large values of t and n this series is very difficult to compute or interpret, but by a similar process to the one outlined for the input indicial admittance h_0, the above power series can be recognized as equivalent to the series expansion of the function $e^{-x}I_n(x)$. Hence we may write

$$h_n(t) = \frac{2}{R}e^{-\frac{2t}{CR}}I_n\left(\frac{2t}{CR}\right)$$

where I_n $(2t/CR)$ is the modified Bessel function of the first kind of order n and argument $(2t/CR)$. By putting $n=0$, the above equation reduces to $h_0(t)$ as it should do. The input current $i_0(t)$ and the output current $i_n(t)$ in response to the direct voltage E_0 shown in Fig. 4, may be obtained from $h_0(t)$ and $h_n(t)$ by multiplying these expressions by the constant E_0. If, however, E_0 is an arbitrary function of the time, these current responses can be obtained by substituting $h_n(t)$ in the superposition theorem expressed by equation (3) and performing the indicated integrations.

ARTIFICIAL COIL-LOADED TELEPHONE LINE

Suppose now that the resistances which constitute the series elements Z_1 of the ladder network are replaced by inductances as shown in Fig. 5.

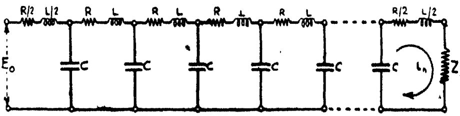

FIG. 5.—Artificial coil-loaded telephone line.

This type of artificial line is closely related, in its properties and performance, to the periodically loaded telephone line, and its solution is, to a first approximation, a working solution for the coil-loaded line invented by Heaviside.

In this case we have for the operational impedances

$$Z_1 = R + pL$$

and

$$Z_2 = 1/pC$$

The power series solution for the current response i_n in the nth section may be developed from equation (6) as before. This equation may be written in the form

$$i_n = \frac{E_0\phi^{-n}(1+\sqrt{1+\phi^{-1}})^{-2n}}{\sqrt{Z_1Z_2}\sqrt{1+\phi}}$$

where $\quad \phi = \left(\frac{p}{\omega_c}\right)^2\left(1 + \frac{R}{pL}\right)$ and $\omega_c = \frac{2}{\sqrt{LC}}$.

$$\therefore \phi^{-n} = \left(\frac{\omega_c}{p}\right)^{2n} \cdot \frac{1}{(1 + R/pL)^n}$$

Also

$$\frac{1}{\sqrt{Z_1 Z_2}} = \sqrt{\frac{C}{L}} \cdot \frac{1}{\sqrt{1 + \frac{R}{pC} \cdot \frac{C}{L}}} = \sqrt{\frac{C}{L}} \cdot \frac{1}{\left(1 + \frac{R}{pL}\right)^{\frac{1}{2}}}.$$

and

$$\frac{1}{\sqrt{1 + \phi}} = \frac{1}{\sqrt{1 + \frac{p^2}{\omega_c^2}\left(1 + \frac{R}{pL}\right)}} = \left(\frac{\omega_c}{p}\right) \cdot \frac{1}{\left(1 + \frac{R}{pL} + \frac{\omega_c^2}{p^2}\right)^{\frac{1}{2}}}$$

Hence we get for the denominator

$$\frac{1}{\sqrt{Z_1 Z_2}\sqrt{1 + \phi}} = \sqrt{\frac{C}{L}}\left(\frac{\omega_c}{p}\right) \cdot \frac{1}{\left(1 + \frac{R}{pL}\right)^{\frac{1}{2}}} \cdot \frac{1}{\left(1 + \frac{R}{pL} + \frac{\omega_c^2}{p^2}\right)^{\frac{1}{2}}}$$

Now

$$\left(1 + \sqrt{1 + \phi^{-1}}\right)^{-2n} = \left(1 + \sqrt{1 + \left(\frac{\omega_c}{p}\right)^2 \cdot \frac{1}{(1 + R/pL)}}\right)^{-2n}$$

Hence the equation for i_n may now be written as

$$i_n = E_0 \sqrt{\frac{C}{L}}\left(\frac{\omega_c}{p}\right)^{2n+1} \cdot \frac{1}{\left(1 + \frac{R}{pL}\right)^{\frac{2n+1}{2}}} \cdot \frac{1}{\left(1 + \frac{R}{pL} + \frac{\omega_c^2}{p^2}\right)^{\frac{1}{2}}} \cdot \left\{1 + \sqrt{1 + \left(\frac{\omega_c}{p}\right)^2 \cdot \frac{1}{\left(1 + \frac{R}{pL}\right)}}\right\}^{-2n}$$

which is a convenient form for asymptotic expansion in inverse powers of p. In order to derive the required power series solution each of the last three factors must be expanded by the binomial theorem and arranged in inverse powers of p. The next step is to multiply the three result-

ing series together and rearrange in inverse powers of p. This series must then be multiplied throughout by the factor $(\omega_c/p)^{2n+1}$ and then the explicit power series solution may be obtained by replacing p^{-n} by $t^n/n!$. This expansion becomes very complicated if a number of terms is required, and it will be found that owing to its slow convergence the series is practically useless to compute from except for small values of the time t associated with the first transient surge. Consequently, it becomes necessary to make an effort to recognize and sum the series, which in this case is a most difficult matter. It will be found that i_n can be written as

$$i_n = E_0\sqrt{\frac{C}{L}}e^{-\psi x}\int_0^x J_{2n}(x_1)I_0(\psi\sqrt{x^2 - x_1^2})dx_1$$

where

$$\psi = \frac{R}{4}\sqrt{\frac{C}{L}}$$

$$x = \omega_c t$$

and J and I are Bessel functions. The evaluation of this expression depends on the numerical computation of Bessel functions and the integration of definite integrals. The latter operation presents no difficulties and the integrations are easily and accurately effected, once the integrands are plotted as time functions, by means of a suitable integraph (or even on the drawing-board). Fortunately, also, the Bessel functions have been the subject of exhaustive study and tabulation. The formal solution of the integral would consist of expanding the Bessel functions as power series, integrating term by term and rearranging. This would yield the formal Heaviside power series solution we have discussed above. It should be noted that the expression for i_n is a special case of a more general formula deduced by Dr. J. R. Carson in his researches on speech-excited transients in transmission networks. For a complete discussion of the artificial coil-loaded line the student is

referred to Carson's paper on 'Transient Oscillations', *Transactions A.I.E.E.*, 1919.

The Heaviside operational scheme we have outlined in this chapter has several distinct advantages from an engineer's point of view. It can be readily carried out to obtain an explicit solution. It involves only elementary mathematics. In many cases the results obtained would be very difficult (or even impossible) to obtain by any other method. The scheme, however, has certain disadvantages. The results obtained are sometimes in a form that is not convenient for numerical study owing to the slow convergence of the series; if, however, the series can be summed in terms of tabulated functions, this difficulty can be avoided.

It must be noted that a power series solution does not exist for all classes of electric circuit problem; for example, in a certain class of network problem relating to transmission lines, a power series solution does not exist. In this class, however, a closely related series in fractional powers of the time t can usually be derived. A discussion of the application of this related series to network problems will be found in Chapter 7.

CHAPTER 5

HEAVISIDE'S LAST THEOREM

IMPULSE FUNCTIONS

IN his researches Heaviside did not bother to construct rigorous mathematical proofs of his theorems; to him a 'rigorous proof' was only an attempt to meet the whims and fancies of certain pure mathematicians. Heaviside 'proved' nothing; he simply made sure he was right. To make sure of being right he appealed to experiment and measurement for a judgement: these practical tests he applied to his theorems in many ways and under many different conditions. When dealing with his asymptotic series, for example, he depended upon a practical grasp of the physical side of the problem he was analysing to guide him away from mathematical pitfalls. In many cases he used physical arguments to establish mathematical results. Thus conflict with pure mathematicians was inevitable. This conflict was unfortunate because it antagonized him, and consequently he made no attempt to present his operational calculus as a rigorous mathematical structure.

In this chapter attention is drawn to an important and far-reaching Heaviside theorem which so far appears to have escaped the notice of engineers. This theorem has been reconstructed from the scattered papers of Heaviside and is probably the last theorem he ever deduced. For this reason it has been called 'Heaviside's Last Theorem'. From the basis of this theorem it is possible to deduce rigorously all the various Heaviside operational processes used in electric circuit theory.

At this stage it would be well to consider the nature of the operator p(or d/dt) which is assumed to act on real

variables. The inverse operator p^{-1} represents the operation $\int dt$. It follows, therefore, that

$$pp^{-1}H(t) = \frac{d}{dt}\int_0^t H(t)\,dt = H(t)$$

and consequently the operator p undoes the operation p^{-1} if p acts after p^{-1}. When p and p^{-1} both occur in an operator the integrations must be carried out before the differentiations.

Otherwise $$p^{-1}pH(t) = \int_0^t H'(t)\,dt = H(t) - H(0)$$

which shows that p and p^{-1} are commutative only if the function operated on vanishes with t, i.e. $H(0) = 0$. It will be noted that Heaviside's unit function [1] preserves this commutative property in the operators which arise in electric circuit theory, because

$$\left.\begin{aligned}[1] &= 1 \text{ for } t \geqslant 0 \\ &= 0 \text{ for } t < 0\end{aligned}\right\}$$

Thus $\frac{\partial}{\partial t}[1]$ is zero everywhere in the range, $-\infty < x < +\infty$, except at the point $x = 0$; at this point the unit function [1] suddenly jumps from zero to unit value as shown in Fig. 2. The derivative of the function is therefore infinite at the point $t = 0$. From the above

$$\begin{aligned}p[1] &= 0 \text{ for } t \gtrless 0 \\ &= \infty \text{ for } t = 0\end{aligned}$$

But $$p^{-1}(p[1]) = p(p^{-1}[1]) = pt = 1$$

Thus $p[1]$ is a function which is everywhere zero, except at the origin where it is infinite; the integral of the function,

however, is unity. This function, which Heaviside called a 'unit impulse', is illustrated in Fig. 6.

To visualize the unit impulse function consider a long thin strip of length l and width δt extending along the axis of y as shown in Fig. 6. Now let $l \rightarrow \infty$ as $\delta t \rightarrow 0$

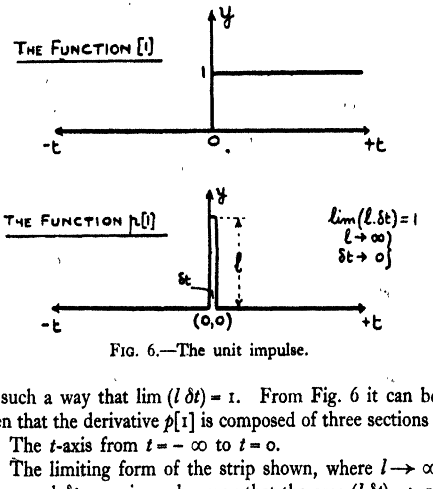

FIG. 6.—The unit impulse.

in such a way that $\lim (l\,\delta t) = 1$. From Fig. 6 it can be seen that the derivative $p[1]$ is composed of three sections:

(1) The t-axis from $t = -\infty$ to $t = 0$.

(2) The limiting form of the strip shown, where $l \rightarrow \infty$ and $\delta t \rightarrow 0$ in such a way that the area $(l\,\delta t) \rightarrow 1$. This strip is perpendicular to the t-axis from the point $t = 0$.

(3) The t-axis from $t = 0$ to $t = +\infty$.

This mathematical idea may be illustrated by considering the charge that passes to the plates of a pure condenser when connected to an e.m.f. Thus if q is the charge, i the current and e the voltage, then

$$q = \int_0^t i\,dt = \int_0^t C\frac{de}{dt}\,dt$$

If C and e have unit values, it follows that

$$q = p^{-1}(p[1]) = 1$$

Thus the current $p\,[1]$ represents a unit impulse, i.e. it is infinite at time $t = 0$ and zero thereafter. This means that at time $t = 0$, when the switch is closed, a unit charge is instantly placed on the plates of the pure condenser.

TRANSFER OPERATORS

The symbolic form of Taylor's theorem was well known in Heaviside's time. It is easy to show that $f\,(x \pm vt)$ is produced by the operator $e^{\pm vt\frac{d}{dx}}$ operating on $f(x)$.

Thus
$$e^{\pm vt\frac{d}{dx}} f(x) = f(x \pm vt)$$

If t is the time and v a velocity, then the expression $f(x - vt)$ represents a curve or wave moving along the axis of x with a velocity v in the positive x direction. In general, the effect of the operator e^{-ph} on $f(t)$ is to translate it bodily through a distance h to the right.

Exponential operators of this type are known as transfer operators and allow functions to be moved about in time or position. Heaviside made great use of these operators

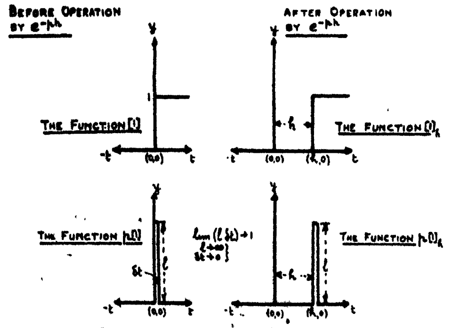

FIG. 7.—Effect of transfer operator.

in the practical applications of his Duplex electromagnetic theory. He used them to remove damping factors; to transform space waves into travelling waves; to make travelling waves stand still so that he could study their distortion with lapse of time.

Now consider the function $e^{-ph}[1]$; the effect of the operator e^{-ph} on $[1]$ is to move the unit function bodily through a distance h to the right as illustrated in Fig. 7. This transferred unit function can be written $[1]_h$; then

$$e^{-ph}[1] = [1]_h$$

Likewise the effect of the operator e^{-ph} on $p[1]$ is to move the unit impulse from the origin to a point on the t-axis where $t = h$. In other words, the effect of the operator is to transfer $p[1]$ with all its properties unchanged to another origin of co-ordinates at the point $(h, 0)$ as shown in Fig. 7.

SPOTTING FUNCTIONS

From Fig. 7 it can be seen that the transfer operator e^{-ph} applied to the unit impulse $p[1]$ generates a function composed of three sections:

(1) The t-axis from $t = -\infty$ to $t = +h$.

(2) The limiting form of the unit area strip extending perpendicular to the t-axis from the point $t = +h$.

(3) The t-axis from $t = +h$ to $t = +\infty$.

Heaviside called this function a 'Spotting function' because it picks out or spots a certain value of a function. In order to illustrate this idea, begin by considering two real variables, such as electric and magnetic forces, which range from 0 to ∞. For simplicity denote these variables by x and t. Let σ denote d/dx and p denote d/dt. Then if $[1]$ represents the unit function, we have

$$\begin{aligned}
\sigma[1] &= \text{unit impulse at the point } x = 0\\
p[1] &= \quad ,, \quad ,, \quad ,, \quad ,, \quad ,, \quad t = 0\\
e^{-\sigma t}\sigma[1] &= \quad ,, \quad ,, \quad ,, \quad ,, \quad ,, \quad x = t\\
e^{-px}p[1] &= \quad ,, \quad ,, \quad ,, \quad ,, \quad ,, \quad t = x
\end{aligned}$$

It is apparent that $e^{-\sigma t}\sigma[1]$ is equal to $e^{-px}p[1]$, because a unit impulse placed at the point $x = t$ on the x axis is the same thing as a unit impulse placed at a point $t = x$ on the t axis. Thus $e^{-\sigma t}\sigma[1]$ is a function of x concentrated at the point $x = t$; while $e^{-px}p[1]$ is a function of t concentrated at the point $t = x$. This fact makes the two spotting functions quantitatively equal; Heaviside demonstrated this by showing that both forms of spotting function can be obtained from the same operational generator, $\sigma p/(\sigma + p)$. By expanding this operator in inverse powers of p and then replacing p^{-n} by $t^n/n!$, the spotting function $e^{-\sigma t}\sigma[1]$ is produced. Likewise, by expanding the operator in inverse powers of σ and then replacing σ^{-n} by $x^n/n!$, the spotting function $e^{-px}p[1]$ is produced. Thus both spotting functions are generated by the same operator and are thus quantitatively equal

$$\frac{\sigma p}{\sigma + p}[1] = e^{-\sigma t}\sigma[1] = e^{-px}p\,[1]$$

From the above it can be seen that a unit impulse is a function which is everywhere zero, except at a single point where it is infinite. The integral of this function, however, is unity.

AN INTEGRAL THEOREM

Consider two real variables x and t ranging from 0 to $+\infty$. Then $f(x)e^{-\sigma t}\sigma[1]$ is zero over all the range $0 < x < +\infty$, except at the point $x = t$. At this point $f(x)e^{-\sigma t}\sigma[1]$ is indeterminate. The integral $\int_{-\infty}^{\infty} f(x)e^{-\sigma t}\sigma[1]dx$ can, however, be evaluated. From Fig. 8 it can be seen that there is no contribution from the range $0 < x < t$, or from the range $t < x < +\infty$, since the integrand is zero over this range. To obtain the contribution at the point $x = t$, consider the function $e^{-\sigma t}\sigma[1]$ as a thin strip of length l and width δx. The area of the strip under the

integrand $f(x)e^{-\sigma t}\sigma[1]$ at the point $x = t$, is therefore of width δx and length $l f(t)$ as illustrated in Fig. 8. In the

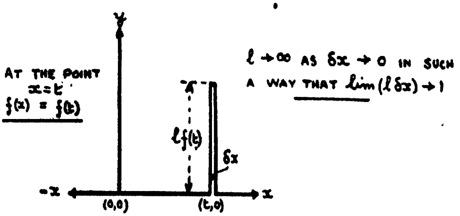

FIG. 8.—The function $y = f(x)e^{-\sigma t}\,\sigma[1]$.

limit $l \to \infty$ as $\delta x \to 0$, in such a way that $\lim\,(l\,\delta x) \to 1$, and $\lim\,\{f(t) l\,\delta x\} \to f(t)$; thus we may write

$$f(t) = \int_{0^-}^{\infty} f(x)e^{-\sigma t}\sigma[1]\,dx$$

which is apparent when we observe that the function to be integrated is an impulse of size $f(x)$ at the point $x = t$, at which point $f(x)$ has the value $f(t)$. This leads us to Heaviside's last theorem which may be stated formally in the following manner. If x and t are two real variables ranging from 0 to ∞, and σ denotes d/dx and p denotes d/dt, then

$$\left.\begin{aligned} f(t) &= \int_0^\infty f(x)\begin{Bmatrix} e^{-\sigma t}\sigma \\ e^{-pxp} \end{Bmatrix} dx \\ \text{and} \quad f(x) &= \int_0^\infty f(t)\begin{Bmatrix} e^{-\sigma t}\sigma \\ e^{-pxp} \end{Bmatrix} dt \end{aligned}\right\} \qquad (7)$$

It will now be shown that Heaviside's last theorem is a fundamental one, i.e. the ideas involved in it are related in a natural way to a large complex of other mathematical

ideas. Thus the theorem leads to many others; not only electric circuit theorems, but also to theorems involving Fourier and Bessel integrals, elliptic functions, &c. In this book, however, we are only interested in its applications to electric circuit theory.

Since Heaviside's time the two most original contributions to electric circuit theory were made by Professor I'A. Bromwich of England and Dr. J. R. Carson of America. Bromwich's contribution will be found in his paper, 'Normal Co-ordinates in Dynamical Systems', read to the London Mathematical Society in 1916. In this paper Bromwich deduces an integral theorem from which it is possible to formulate an explicit relationship between the indicial admittance function and the impedance function. Bromwich's discussion of Heaviside's methods by means of contour integrals in the complex plane is one of great beauty and power. It justifies and extends many of Heaviside's processes. Nevertheless, it would be difficult to imagine two methods more unlike than those of Bromwich and Heaviside. In this connexion it is interesting to note that Bromwich has repeatedly declared that the direct operational method of Heaviside is far the best for dealing with the class of problems concerned. Carson's method, on the other hand, is very similar in form to that of Heaviside. It approaches the subject in an original manner, and, by basing the analysis on what is known as an integral equation it puts to new use a large number of operational results in such a way as to greatly extend their range of application. Carson's contribution to electric circuit theory will be found in his book, *Electric Circuit Theory and the Operational Calculus*, published in 1926. In this book Carson deduces an infinite integral theorem which expresses an implicit relationship between the indicial admittance function and the impedance function. From this theorem Carson establishes the operational formulae given by Heaviside and presents the operational calculus as a rigorous structure. In his book, however,

Carson has failed to notice that his fundamental integral equation is only a special case of Heaviside's last theorem and that Bromwich's contour integral is the solution of his equation.

Heaviside gave the Carson theorem on page 236 of the third volume of his *Electromagnetic Theory*, published in 1912; his notation and discussion at this point, however, was so different from his usual scheme that the theorem is not apparent until it has been transformed into a more familiar notation. Heaviside did not use the theorem extensively for electric circuit analysis; unlike Carson, he did not use it to establish or transform operational formulae, but used it to evaluate infinite integrals. This may be due to the fact that he deduced the theorem rather late in life, or he may have preferred other methods. In this connexion it must be noted that there are many perfectly valid methods of establishing operational formulae without using the integral theorem, for example, the expansion theorem and power series methods discussed in Chapters 2 and 3. Consequently the use Heaviside made of the theorem, namely, to evaluate the infinite integrals which arise in electromagnetic theory and which cannot be evaluated readily by any other means, may become the more valuable process as the operational calculus develops. In any case the theorem is of use either way. Since our subject is the analysis of electric circuits, and not the solution of infinite integrals, we return to a discussion of Carson's integral theorem.

CARSON'S INTEGRAL EQUATION

Applying equation (7) to electric circuit theory, we obtain

$$\frac{1}{Z(p)} = \int_0^\infty h(x)e^{-px}p\,dx$$

by replacing $h(t)$ by its generating impedance function

$1/Z(p)$. Since the integration is with respect to x, the operator p that occurs can be treated exactly as though it were a parameter; and in fact it can be replaced by a parameter λ. Thus we obtain on replacing x by t

$$\frac{1}{Z(\lambda)} = \lambda \int_0^\infty h(t) e^{-\lambda t}\, dt. \quad . \quad . \quad . \quad . \quad (8)$$

which is Carson's integral equation. It can be seen that this equation expresses an implicit relationship between the impedance function and the indicial admittance function $h(t)$. By virtue of this relationship Carson puts to new use a large number of established operational solutions in such a way as greatly to extend their range of application. Moreover, Carson introduced Borel's theorem into the analysis and thereby gave the method much additional power.

The formal solution of Carson's integral equation is given by the Bromwich integral, where

$$h(t) = \frac{1}{2\pi j} \int_{C-j\infty}^{C+j\infty} \frac{e^{\lambda t}}{\lambda Z(\lambda)}\, d\lambda \quad . \quad . \quad . \quad . \quad (9)$$

which expresses an explicit relationship between the indicial admittance and the impedance function. The Bromwich solution is discussed in detail in Chapter 8. For the present it can be noted that the above equations are valid for all values of λ, for which the real part is greater than some finite constant C. This constant C must be at least large enough to make the infinite integrals converge. For the majority of passive networks this constant may be taken as 0; if the network employs some form of feedback, however, the above equations are valid only when C is greater than some finite value.

The foregoing analysis naturally suggests that the methods of Carson and Bromwich are closely related to Heaviside's last theorem. It also follows that a Heaviside

operational equation is a shorthand way of writing down infinite integral equations or contour integrals. If this fact is kept in mind the operator p will lose its mysterious character: thus if we try to evaluate the operational equation $Z(p) = \sqrt{p}[1]$, it does not mean that we are seeking to find the square root of the differential coefficient of a discontinuous function, which is meaningless, but simply that we have written down in shorthand the Carson integral equation,

$$\frac{1}{\sqrt{\lambda}} = \lambda \int_0^\infty h(t) e^{-\lambda t}\, dt,$$

or the Bromwich contour integral

$$h(t) = \frac{1}{2\pi j} \int_{C-j\infty}^{C+j\infty} \frac{e^{\lambda t}}{\lambda\sqrt{\lambda}}\, d\lambda$$

which shows that the operator p has lost its original significance and is now the complex argument of functions which obey all the laws of mathematical analysis.

CHAPTER 6

THE ESTABLISHMENT OF OPERATIONAL FORMULAE

SURVEY OF FUNDAMENTAL FORMULAE

IN Chapter 1 the analysis of a circuit problem was defined as the determination of the variation of a physical magnitude in terms of time and some dimension; more precisely, the determination of the oscillations of a linearly connected system described by a set of differential equations with constant coefficients or a partial differential equation of the wave motion type. It was shown that these descriptive differential equations may be readily written down by an application of Kirchhoff's laws to the problem. The electric network considered was assumed to be in a passive state when at reference time $t = 0$, a unit electromotive force [1], was suddenly applied. The first step in the operational analysis of a network problem is the purely formal one of replacing the differential operator d/dt by the symbol p, and thereby reducing the differential equations to an algebraic form. From the rules of algebra we obtain the Heaviside operational equation which may be written formally as

$$h(t) = \frac{1}{Z(p)} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This equation is a purely symbolic expression and its interpretation means a transformation to yield a function of the time t no longer involving the differential operator p. In this expression $h(t)$ represents the indicial admittance function, while $Z(p)$ is the impedance function and is the means whereby the parameters and the connexions of the network enter the mathematical formulation. Thus $Z(p)$ completely specifies the network; if p is replaced by $j\omega$,

where $\omega/2\pi$ is the frequency, then $Z(j\omega)$ is the well-known steady-state impedance function of alternating current theory. From Chapter 1 it will be recalled that the Heaviside unit discontinuous function [1] should appear as the operand on both sides of equation (4); but since this unit function is always tacitly assumed to apply to all the operational equations which arise in electric circuit theory it need not be explicitly stated.

In Chapter 5 it was pointed out that an operational equation is not a true algebraic equation and has no literal meaning by itself; it is simply an easy shorthand way of writing down a set of descriptive differential equations which have been obtained by applying physical laws to the network. It was also shown as a corollary to Heaviside's last theorem that to every operational equation (4) there corresponds the Carson integral equation

$$\frac{1}{Z(\lambda)} = \lambda \int_0^\infty e^{-\lambda t} h(t)\, dt \quad . \quad . \quad . \quad . \quad (8)$$

which expresses an implicit relationship between $h(t)$ and $Z(p)$. By virtue of this relationship Carson has shown how to extend the range of operational equations and ratify unestablished operational results.

The formal solution of Carson's integral equation is given by the Bromwich contour integral, where

$$h(t) = \frac{1}{2\pi j} \int_{C-j\infty}^{C+j\infty} \frac{e^{\lambda t}}{\lambda Z(\lambda)}\, d\lambda \quad . \quad . \quad . \quad . \quad (9)$$

which expresses an explicit relationship between $h(t)$ and $Z(p)$. By virtue of this relationship Bromwich has shown how to evaluate Heaviside operational equations by direct integration and thereby determine $h(t)$.

Having determined $h(t)$, the response $i(t)$ of the network to any arbitrary applied electromotive force $E(t)$ is obtained

from an application of the superposition theorem

$$i(t) = \frac{d}{dt}\int_0^t E(t - \psi)h(\psi)\, d\psi \quad . \quad . \quad . \quad (3)$$

These equations constitute a complete mathematical formulation of the problem of electric circuit analysis; and their solution automatically takes care of the necessary terminal conditions. That is to say, contrary to the classic method of solution discussed in Chapter 1, the explicit determination of roots of high degree algebraic equations and the explicit determination of integration constants is rendered unnecessary.

It will now be apparent the expansion theorem and power series solution discussed in Chapters 2, 3 and 4 are examples of the fact that if an operational equation can be expanded in a sum of terms, thus

$$\frac{1}{Z(p)} = \frac{1}{Z_1(p)} + \frac{1}{Z_2(p)} + \ . \ . \ . + \frac{1}{Z_n(p)}$$

and if the auxiliary operational equations

$$h_1 = \frac{1}{Z_1(p)}, \quad h_2 = \frac{1}{Z_2(p)}, \ . \ . \ . \ h_n = \frac{1}{Z_n(p)}$$

can be solved, then

$$h(t) = h_1 + h_2 + h_3 + \ . \ . \ . + h_n$$

Thus we can juggle the left-hand side of Carson's integral equation (8) in any way we please provided the changed expression is actually equivalent to $1/Z(\lambda)$. We may develop in series, split into partial fractions or the like, all without altering the validity of Carson's integral equation. Thus an operator may be expanded into a sum of terms and to each separate term will correspond a valid integral of the Carson type.

SHORT TABLE OF CARSON INTEGRAL EQUATIONS

A large number of infinite integrals of the Carson type have been worked out by mathematicians. Consequently

the solutions of many of the operational equations which arise in electric circuit theory can frequently be written down by inspection. When this is not the case, however, the proper procedure is then to expand the function $1/Z(\lambda)$ by any means, in such a form that the individual terms are recognizable as being symbolically equivalent with Carson integral equations. It must be noted that not all the operational equations that arise in electric circuit theory can be recognized without transformation; this idea corresponds with the fact that a table of integrals is not always sufficient to recognize an integral, but must be supplemented by general methods of transformation and integration. Below is a small list of some Carson integral equations. This list could be greatly extended. Even this short list, however, contains the solutions of many operational equations which occur in important technical problems.

$$\frac{\lambda}{(\lambda + b)^{n+1}} = \lambda\int_0^\infty e^{-\lambda t}\cdot\frac{t^n}{n!}\cdot e^{-bt}\cdot dt \quad . \quad . \quad . \quad (A)$$

$$\sqrt{\lambda}\cdot e^{-2\sqrt{\lambda b}} = \lambda\int_0^\infty e^{-\lambda t}\cdot\frac{e^{-b/t}}{\sqrt{\pi t}}\cdot dt \quad . \quad . \quad . \quad (B)$$

$$\frac{\lambda}{r}\left(\frac{r - \lambda}{b}\right)^n = \lambda\int_0^\infty e^{-\lambda t}J_n(bt)\,dt \quad . \quad . \quad . \quad (C)$$

where $r^2 = \lambda^2 + b^2$

$$\frac{\lambda}{\sqrt{\lambda^2 + 1}}\, e^{-b\sqrt{\lambda^2+1}} = \lambda\int_0^\infty e^{-\lambda t}J_0(\sqrt{t^2 - b^2})\,dt \quad . \quad (D)$$

where $t \geqslant b$

$$\frac{1}{\sqrt{1 + \frac{2b}{\lambda}}} = \lambda\int_0^\infty e^{-\lambda t}\cdot e^{-bt}I_0(bt)\,dt \quad . \quad . \quad . \quad (E)$$

where J_n and I_n are Bessel functions and b a real quantity.

DEVELOPMENT OF INTEGRAL SOLUTIONS BY OPERATIONAL METHODS

We will now discuss in greater detail how Carson's integral solutions may be built up from the operational methods discussed in Chapters 2 and 3. Consider the first solution (A); putting $n = 0$, we have

$$\frac{\lambda}{\lambda + b} = \lambda \int_0^\infty e^{-\lambda t} . e^{-bt} . dt$$

from which we obtain the operational solution

$$\frac{p}{p+b} = e^{-bt}$$

which can be easily verified by an application of Heaviside's expansion theorem. In order to obtain the operational solution of $\frac{1}{p+b}$, from the above equation, write $\frac{1}{p+b}$ as $\frac{1}{p} . \frac{p}{p+b}$. We have by associating p^{-1} with $\int_0^t dt$, the solution

$$\frac{1}{p+b} = \int_0^t e^{-bt} dt = \frac{1}{b}(1 - e^{-bt})$$

This operational solution is a well-known result, but the method of writing the operational equation as the product of two operators and then evaluating should be carefully noted.

Suppose we transform an operational equation into the form of the product of two operators, namely, $\frac{1}{Z_1(p)}\left\{\frac{1}{Z_2(p)}\right\}$ and write the results of the individual operators as $h_1(t) = 1/Z_1(p)$ and $h_2(t) = 1/Z_2(p)$. Then we may write the operational product as

$$\frac{1}{Z_1(p)} . \left\{\frac{1}{Z_2(p)}\right\} = \frac{1}{Z_1(p)} . h_2(t)$$

Now this equation may be regarded as the effect of suddenly applying an electromotive force $h_2(t)$ to a network the impedance function of which is $Z_1(p)$. Consequently the response of the network may be obtained from the superposition theorem expressed by equation (3). Thus we may write

$$\frac{1}{Z_1(p)}\cdot h_2(t) = \frac{d}{dt}\int_0^t h_1(\psi)h_2(t-\psi)\,d\psi$$

It will be noted that this expression involves a variable parameter under the sign of integration and that ψ is the variable of integration which disappears when the integration is performed. The expansion and evaluation of this type of integral is discussed in Chapter 8 where it will be shown that the integral expression above can be written in the expanded form

$$\frac{1}{Z_1(p)}\cdot h_2(t) = h_1(t)h_0(0) + \int_0^t h_1(t-\psi)h_2^1(\psi)d\psi$$

Since we have evaluated $1/(p+b)$ and $p/(p+b)$, it will be observed that this expansion will enable us to evaluate operational expressions like $p/(p+b)^{n+1}$, where n is any positive integer, and consequently establish Carson's integral equation (A).

Let $n=1$, then we have

$$\frac{p}{(p+b)^2} = \frac{1}{p+b}\cdot\frac{p}{p+b}$$

where $h_1(t) = \frac{1}{b}(1-e^{-bt})$ and $h_2(t) = e^{-bt}$. Inserting these results in the expansion of $h_2(t)/Z(p)$, and performing the indicated integrations, we find that

$$\frac{p}{(p+b)^2} = te^{-bt}$$

which agrees with the Carson equation (A) for $n=1$. We

can continue the process by evaluating the operational expression $p/(p+b)^3$, and show that we get the same result as would be obtained by putting $n = 2$, in the integral equation (A). Writing the operational expression in the form

$$\frac{1}{p+b}\cdot\frac{p}{(p+b)^2}$$

we have $\quad h_1(t) = \frac{1}{b}(1 - e^{-bt})$ and $h_2(t) = te^{-bt}$

Inserting these results in the expansion of $h_2(t)/Z_1(p)$, and performing the indicated operations, we find that

$$\frac{p}{(p+b)^3} = \frac{t^2}{2!}e^{-bt}$$

which agrees with the result obtained from equation (A) when $n = 2$.

Continuing this process, we obtain the general operational solution

$$\frac{p}{(p+b)^n} = \frac{t^{n-1}}{(n-1)!}e^{-bt}$$

If the operator p is replaced by the parameter λ, this solution may be written as the infinite integral

$$\frac{\lambda}{(\lambda+b)^{n+1}} = \lambda\int_0^\infty e^{-\lambda t}.\frac{t^n}{n!}.e^{-bt}.dt$$

which is a convenient form for transformation. Putting $b = 0$, in this expression, we get the well-known result

$$p^{-n} = \frac{t^n}{n!}$$

The above shows how the product of two operators may be transformed by means of the superposition theorem. It must be noted, however, that it is possible to make these transformations in a much easier manner.

BOREL'S THEOREM

There is a theorem due to Borel which is more convenient for dealing with the product of operators than the Superposition theorem. We begin our discussion of Borel's theorem by expressing it in the terminology of this book.

Suppose that the function $1/Z(p)$ can be transformed into the form

$$\frac{1}{Z(p)} = \frac{1}{p}\cdot\frac{1}{Z_1(p)}\cdot\frac{1}{Z_2(p)}$$

where $h_1(t) = 1/Z_1(p)$, $h_2(t) = 1/Z_2(p)$, and p^{-1} denotes the operation $\int_0^t dt$.

Then from Borel's theorem we may write

$$h(t) = \int_0^t h_1(\psi)h_2(t-\psi)\,d\psi$$

$$= \int_0^t h_1(t-\psi)h_2(\psi)\,d\psi$$

where ψ is the variable of integration.

To illustrate the use of this theorem, let us evaluate again the operational expression $p/(p+b)^3$. In this case we write

$$\frac{p}{(p+b)^3} = \frac{1}{p}\cdot\frac{p}{p+b}\cdot\frac{p}{(p+b)^2}$$

where $h_1(t) = e^{-bt}$ and $h_2(t) = te^{-bt}$

Substituting in Borel's theorem, we get

$$h(t) = \int_0^t e^{-b\psi}(t-\psi)e^{-b(t-\psi)}\,d\psi$$

which reduces to $t^2e^{-bt}/2!$, as before. It should be noted

that it is much simpler to use this method than the superposition theorem.

We have discussed in some detail certain operational processes which lead to the formulation of integral equations of the Carson type. These infinite integrals should be carefully tabulated. Thus when a new operational expression is met it may be interpreted at once by reference to the table. If, however, this cannot be done directly, it will be necessary to pick the most likely-looking infinite integral and transform it into the required form. An illustration of the methods of transforming these infinite integrals will be given in the next chapter.

HEAVISIDE'S 'SHIFTING' TRANSFORMATION

In order to reduce the work involved in expanding operational expressions containing exponential damping or attenuation factors, Heaviside introduced what he called a 'shifting' transformation. This transformation consisted in making certain changes in the operator and writing the exponential factor outside; these changes generally resulted in simplifying the operator and thereby making it easier to interpret. We will describe this process by means of the Carson integral. It has been shown that if the operational equation

$$h(t) = \frac{1}{Z(p)}$$

exists, then

$$\frac{1}{Z(\lambda)} = \lambda \int_0^\infty e^{-\lambda t} h(t)\, dt$$

where λ is a parameter. Now replace λ by $(\lambda + \alpha)$, we get

$$\frac{1}{Z(\lambda + \alpha)} = (\lambda + \alpha) \int_0^\infty e^{-\lambda t}[e^{-\alpha t} h(t)]\, dt$$

which may be written in the form

$$\frac{1}{1+\frac{\alpha}{\lambda}} \cdot \frac{1}{Z(\lambda+\alpha)} = \lambda \int_0^\infty e^{-\lambda t}[e^{-\alpha t}h(t)]\,dt$$

from which we obtain the operational equation

$$e^{-\alpha t}h(t) = \frac{p}{p+\alpha} \cdot \frac{1}{Z(p+\alpha)}$$

Replacing $h(t)$ by its operational generator $1/Z(p)$, and changing the sign of α, we obtain

$$\frac{1}{Z(p)} = e^{-\alpha t} \cdot \frac{p}{p-\alpha} \cdot \frac{1}{Z(p-\alpha)} \quad . \quad . \quad . \quad . \quad (10)$$

This transformation enables us, when we have an operational expression, one factor of the result of which we suspect to be exponential, to write down that part immediately and make the corresponding changes in the operational equation. Formulae for shifting other functions may also be developed.

As a simple example illustrating the application of this exponential transformation, consider a circuit consisting of a resistance R, an inductance L and a capacity C in series, to which the unit electromotive force [1] is suddenly applied. The operational equation is

$$h(t) = \frac{1}{R + pL + 1/pC}$$

$$= \frac{p}{L(p-\lambda_1)(p-\lambda_2)}$$

where $\quad \lambda_1 = -\alpha + j\beta$

$\quad \lambda_2 = -\alpha - j\beta$

and $\quad \alpha = R/2L$

$$\beta = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

Thus $h(t)$ may be written in the form

$$h(t) = \frac{1}{\beta L} \cdot \frac{p\beta}{(p+\alpha)^2 + \beta^2}$$

We suspect the presence of the factor $e^{-\alpha t}$, and using the Heaviside shifting transformation, we write

$$h(t) = \frac{1}{\beta L} \cdot e^{-\alpha t} \cdot \frac{p}{p-\alpha} \cdot \frac{(p-\alpha)\beta}{(p-\alpha+\alpha)^2 + \beta^2}$$

$$\therefore\ h(t) = \frac{1}{\beta L} \cdot e^{-\alpha t} \cdot \frac{p\beta}{p^2 + \beta^2}$$

It is a simple matter to evaluate the operational part of this expression. Expanding in inverse powers of p and then replacing p^{-n} by $t^n/n!$, we find that

$$\frac{p\beta}{p^2 + \beta^2} = \sin \beta t$$

Thus we may write

$$h(t) = \frac{1}{\beta L} \cdot e^{-\alpha t} \sin \beta t$$

This result is for the case where $\frac{1}{LC} > \alpha^2$. If, however, $\alpha^2 > \frac{1}{LC}$, we note that β now becomes a pure imaginary. Consequently we put $\beta = j\beta_1$, in the above equation, and obtain

$$h(t) = \frac{1}{j\beta_1 L} e^{-\alpha t} \sin j\beta_1 t$$

Since $\sin j\beta_1 t = j \sinh \beta_1 t$, we obtain

$$h(t) = \frac{1}{\beta_1 L} e^{-\alpha t} \sinh \beta_1 t$$

These results are in agreement with those obtained by the expansion theorem in Chapter 2. The ease with which the shift transformation enables these expressions to be arrived at should be noted.

FRACTIONAL ORDER DERIVATIVES

In Chapter 3 it was shown that the initial response of an electric network may be obtained by putting $t = 0$ in the indicial admittance function $h(t)$. Consequently the function $h(0)$ represents the distribution of current in the network at the instant the switch is closed. For all practical networks met with in practice $h(0) = 0$, because inductances will carry no current and condensers will support no voltage until a finite time has elapsed. The same effect may be obtained by considering the network response to an infinite frequency because for this limiting frequency the inductances will carry no current and the condensers will support no voltage. Thus if $Z(\infty)$ represents the steady-state impedance function for an infinite frequency, we may write

$$\frac{1}{Z(\infty)} = h(0) = 0$$

because for all practical networks $Z(\infty) = \infty$.

Let us now consider some case in which $Z(\infty)$ is either finite or zero. For a pure condenser, $Z(j\omega) = 1/j\omega C$, and for an infinite frequency $Z(\infty) = 0$. Thus it is only possible to obtain such a result by applying a voltage to a condenser which has neither resistance nor inductance. Since it is impossible to build a condenser entirely devoid of resistance and inductance, it follows that the case where $Z(\infty)$ comes out zero is a mathematical fiction. No matter how small this inherent resistance and inductance may be, it will always be sufficient to eliminate the difficulties associated with infinite currents at time $t = 0$. This follows because $R + j\omega L$ must always be added to $1/j\omega C$, and consequently $Z(\infty)$ is now infinite, however small R and L can be made.

Because the condition where $h(0) = \infty$ cannot exist physically, it must not be thought that this case is without theoretical importance. It will be recalled that in Chapter 5 we showed how Heaviside reasoning in terms of impulses associated with infinite initial currents established many

important mathematical results. We discussed the case of impulsive response in a lumped system in which $Z(\infty) = 0$. We proceed now to the case of impulsive response in a distributed system with the object of introducing the reader to Heaviside's fractional order derivative operators. Let it be required to determine the current i_0 entering a non-inductive cable. Now a non-inductive cable has a distributed series resistance R and shunt capacity C per unit length, $i_0 = V_0/Z_0$, where V_0 is the applied voltage and Z_0 the characteristic impedance. This impedance is given by the expression $\sqrt{R/j\omega C}$. In order to obtain the operational equation, put V_0 equal to the unit function [1] and replace $j\omega$ by p in Z_0. By making C numerically equal to R, we obtain the operational equation

$$h(t) = \sqrt{p}[1]$$

The question now arises as to what meaning can be given to this operational expression. We have shown that an operational equation is a shorthand statement of an analytic equation. In this analytic equation the operator p has lost its original significance and has become the complex argument λ of a function obeying all the laws of analysis. Hence from Carson's integral theorem we may write

$$\sqrt{\lambda} = \lambda \int_0^\infty e^{-\lambda t} h(t)\, dt$$

The value of $h(t)$ may be determined in a number of different ways. Perhaps the simplest is to refer to a table of Carson integrals and pick out the solution. From the short table given in this chapter we cannot find the exact form, but formula (B) resembles it. If we put $b = 0$ in formula (B), we have the required form, namely

$$\sqrt{\lambda} = \lambda \int_0^\infty e^{-\lambda t} . \frac{1}{\sqrt{\pi t}} . dt$$

from which we obtain the result

$$h(t) = \sqrt{p} = \frac{I}{\sqrt{\pi t}}$$

This result is an important one and may be easily verified by direct integration of the Carson integral. By putting $t = \omega^2$ in this integral we obtain

$$\sqrt{\lambda} = \frac{2\lambda}{\sqrt{\pi}} \int_0^\infty e^{-\omega^2\lambda}\, d\omega$$

Now put $x = \omega\sqrt{\lambda}$, and we obtain

$$\sqrt{\lambda} = 2\sqrt{\frac{\lambda}{\pi}} \int_0^\infty e^{-x^2}\, dx$$

Using the well-known result, $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$, it can be seen that the Carson integral reduces to an identity. Thus the interpretation of $\sqrt{p}$ as $1/\sqrt{\pi t}$ is rigorously established.

From this interpretation of $\sqrt{p}$ it can be seen that the entering current $h(t)$ has an infinite value at time $t = 0$, and then dies away in accordance with the formula $1/\sqrt{\pi t}$. This infinite initial current results from our ignoring the distributed inductance of the cable, which, no matter how small, keeps the initial input current finite. Now the charge q_0 that enters this cable may be readily obtained by operating on $h(t)$ by p^{-1}.
We have

$$q_0 = \frac{1}{p} \cdot \frac{1}{\sqrt{\pi t}}$$

Associating p^{-1} with $\int_0^t dt$ we obtain the explicit solution,

$$q_0 = \int_0^t \frac{1}{\sqrt{\pi}}.t^{-\frac{1}{2}}.dt = \frac{2}{\sqrt{\pi}}.\sqrt{t}$$

It can be seen from this expression that the charge that enters the cable is finite for all positive values of t, although $h(t)$ is infinite at time $t = 0$. Thus the operator for q_0 is $\frac{1}{p}\sqrt{p}$ or $1/\sqrt{p}$, which must be interpreted as $2\sqrt{\frac{t}{\pi}}$.

When Heaviside encountered operational expressions like $\sqrt{p}$ for the first time in his transmission researches, his method of interpretation was very different from the one outlined above. In his early researches Heaviside made no pretence to rigorous formulation and allowed only physical and vectorial ideas to guide his mathematics. One of his methods was to obtain the explicit solution of a number of problems by classical methods and then compare them directly with their operational equations. Thus he was led to attach a definite meaning to his operators, which significance he would apply to those problems in which the classical methods failed him. An example will illustrate this kind of inductive reasoning. The artificial non-inductive telegraph cable discussed in Chapter 4 can be made to simulate the corresponding smooth cable by making the number of sections n very large and the C and R elements very small, and then proceed to the limit. It was shown in Chapter 4 that the input current $h_0(t)$ of the artificial telegraph line is

$$h_0(t) = \frac{2}{R}e^{-\frac{2t}{CR}}I_0\left(\frac{2t}{CR}\right)$$

When the number of sections is made very large and the C and R elements very small, the product CR becomes an

increasingly small quantity, and for values of $t > 0$, the argument $(2t/CR)$ of the I_0 Bessel function becomes increasingly large. Thus we may use the approximation, $\exp\left(\frac{2t}{CR}\right)\Big/\sqrt{2\pi\left(\frac{2t}{CR}\right)}$, for the I_0 Bessel function. Then the input current for the artificial cable reduces to

$$h_0(t) = \sqrt{\frac{C}{R}} \cdot \frac{1}{\sqrt{\pi t}}$$

Equating this to the operational expression for the input current for the corresponding smooth line, we have, when C is made numerically equal to R, $\sqrt{p} = \frac{1}{\sqrt{\pi t}}$, as before. The importance of these fractional order operators in transmission line theory is discussed in Chapter 7.

CHAPTER 7

TRANSMISSION LINES

TRANSFORMATION OF AN INTEGRAL EQUATION

CONSIDER a uniform transmission line having a distributed resistance R, a series inductance L, and a shunt capacity C, each per mile of length. For the moment we will neglect the effect of the distributed leakance G. In order to compute the characteristics of the travelling waves on such a system we must first formulate the indicial admittance function $h(t)$.

From alternating current steady-state theory we have

$$i = \frac{V_0}{Z_0} e^{-Px}$$

where i represents the current x miles from the sending end of a uniform transmission line, of propagation constant P, terminated with its characteristic impedance Z_0. In this expression V_0 denotes the sending-end voltage of frequency $\omega/2\pi$, and

$$P = \sqrt{j\omega C(R + j\omega L)}$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{j\omega C}}$$

In order to formulate the operational equation for $h(t)$, put $V_0 = [1]$ and $j\omega = p$. Then we obtain

$$h(t) = \sqrt{\frac{pC}{R + pL}} . e^{-x\sqrt{pC(R+pL)}}$$

This operational equation may be written in the more convenient form

$$h(t) = \sqrt{\frac{C}{L}} . \frac{p}{\sqrt{p^2 + 2\phi p}} . e^{-\frac{x}{v}\sqrt{p^2+2\phi p}}$$

where $v = 1/\sqrt{LC}$, and $\phi = R/2L$

Replacing the differential operator p by the parameter λ, we may write the Carson integral equation as

$$\left.\begin{aligned} \frac{\lambda e^{-\frac{x}{v}\sqrt{\lambda^2+2\phi\lambda}}}{\sqrt{\lambda^2+2\phi\lambda}} &= \lambda\int_0^\infty e^{-\lambda t}h_0(t)\,dt \\ \text{where} \qquad h(t) &= \sqrt{\frac{C}{L}}.h_0(t) \end{aligned}\right\} \quad . \quad . \quad . \quad (11)$$

The problem is thus reduced to evaluating the function $h_0(t)$ from the integral equation. The first step in finding a solution is to search through a table of Carson integrals for the required solution. If the required solution cannot be found in the tables it will be necessary to pick a solution that resembles the one sought and then transform it into the required form. Searching through the short table given in Chapter 6, we find that the required solution is not given; but the integral nearest the required form is found to be

$$\frac{\lambda e^{-b\sqrt{\lambda^2+1}}}{\sqrt{\lambda^2+1}} = \lambda\int_0^\infty e^{-\lambda t}J_0(\sqrt{t^2-b^2})\,dt \quad . \quad . \quad (12)$$

where J_0 is a Bessel function of the first kind and $t \geqslant b$. In the integral the lower limit of zero may be replaced by b, since the Bessel function is zero for $t < b$, and exists only when $t \geqslant b$. It can be seen that the integral (12) resembles the integral equation (11) to some extent, and this resemblance suggests that we transform (12) in such a way that it will yield the function $h_0(t)$.

Begin this transformation by putting $Z = b\lambda$ and $t = bt_1$ in equation (12). Making these substitutions, we obtain

$$\frac{e^{-\sqrt{\lambda^2+b^2}}}{\sqrt{\frac{z^2}{b^2}+1}} = \int_1^\infty e^{-\frac{Z}{b}.bt_1}J_0(b\sqrt{t_1^2-1})b\,dt_1$$

In order to preserve our original notation in the parameter λ we replace Z by λ and t_1 by t.

Now put $\lambda = \omega + h$, and obtain

$$\frac{e^{-\sqrt{(\omega+h)^2+b^2}}}{\sqrt{(\omega+h)^2+b^2}} = \int_1^\infty e^{-\omega t}.e^{-ht}J_0(b\sqrt{t^2-1})\,dt$$

Replacing ω by λ, and then making the substitutions $\lambda = \frac{x}{v}q$ and $t_2 = \frac{x}{v}t$, we obtain

$$\frac{e^{-\sqrt{\left(\frac{x}{v}q+h\right)^2+b^2}}}{\sqrt{\left(\frac{x}{v}q+h\right)^2+b^2}} = \int_{\frac{x}{v}}^\infty e^{-qt_2}.e^{-\frac{vh}{x}t_2}J_0\left(b\sqrt{\frac{v^2}{x^2}t_2^2-1}\right)\frac{v}{x}\,dt_2$$

which may be written as

$$\frac{e^{-\frac{x}{v}\sqrt{(q+h_1)^2+b_1^2}}}{\sqrt{(q+h_1)^2+b_1^2}} = \int_{\frac{x}{v}}^\infty e^{-qt_2}.e^{-h_1t_2}J_0\left(b_1\sqrt{t_2^2-\frac{x^2}{v^2}}\right)dt_2$$

where $h_1 = \frac{v}{x}h$ and $b_1 = \frac{v}{x}b$. In order to keep to our original notation we replace q by λ and t_2 by t. Thus we obtain the Carson integral solution

$$\frac{\lambda e^{-\frac{x}{v}\sqrt{(\lambda+h_1)^2+b_1^2}}}{\sqrt{(\lambda+h_1)^2+b_1^2}} = \lambda\int_{\frac{x}{v}}^\infty e^{-\lambda t}.e^{-h_1t}J_0\left(b_1\sqrt{t^2-\frac{x^2}{v^2}}\right)dt$$

Comparing this solution with equation (11), it can be seen that they can be made identical by putting $h_1 = \phi$ and $b_1 = j\phi$. Consequently the required solution may now be written as

$$\frac{\lambda e^{-\frac{x}{v}\sqrt{\lambda^2+2\phi\lambda}}}{\sqrt{\lambda^2+2\phi\lambda}} = \lambda\int_{\frac{x}{v}}^\infty e^{-\lambda t}.e^{-\phi t}I_0\left(\phi\sqrt{t^2-\frac{x^2}{v^2}}\right)dt$$

from which it can be seen that the function $h_0(t)$ of equation (11) is

$$h_0(t) = e^{-\phi t} I_0\left(\phi\sqrt{t^2 - \frac{x^2}{v^2}}\right)$$

In this equation $I_0(x) = J_0(jx)$, and is a Bessel function of the first kind with imaginary argument and of order zero. Consequently it follows that the indicial admittance function $h(t)$ and the solution of the operational equation may be written as

$$\left.\begin{aligned} h(t) &= \sqrt{\frac{C}{L}} \cdot \frac{p}{\sqrt{p^2 + 2\phi p}} \cdot e^{-\frac{x}{v}\sqrt{p^2+2\phi p}} \\ &= \sqrt{\frac{C}{L}} \cdot e^{-\phi t} I_0\left(\phi\sqrt{t^2 - \frac{x^2}{v^2}}\right) \end{aligned}\right\} \quad . \quad (14)$$

where $v = 1/\sqrt{LC}$ and $\phi = R/2L$.

CHARACTERISTICS OF THE WAVE 'HEAD'

Consider a point x miles from the sending end of the transmission line. From equation (14) it can be seen that $h(t)$ is zero until $t = x/v$, at which instant it jumps suddenly to the value H, where

$$H = \sqrt{\frac{C}{L}} \cdot e^{-\frac{\phi x}{v}} = \sqrt{\frac{C}{L}} \cdot e^{-\frac{1}{2}\sqrt{\frac{CRx^2}{L/R}}}$$

After this initial jump $h(t)$ may attain a maximum value which is large compared with the 'head' H of the travelling wave as shown in Fig. 9.

As the length of the line x and the value of $-\frac{Rx}{2}\sqrt{\frac{C}{L}}$ increases, the relative magnitude of the 'tail' as compared with the 'head' of the wave increases. When the length of the line becomes very great, or the value of L is decreased and becomes too small to be effective, the head H of the wave becomes negligibly small, and the wave, except in the

neighbourhood of its minute head, corresponds to the wave in the non-inductive ($L = 0$) line as shown in Fig. 9.

The exponential factor in the equation for H is the most important, and in order to make the head of the wave as great as possible, the inductive time constant L/R should

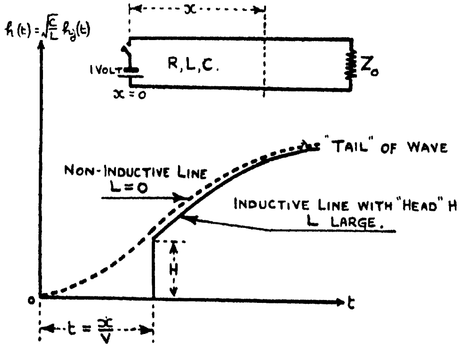

FIG. 9.—Effect of inductance L.

be made as great as possible in comparison with the capacity time constant CRx^2. It can be seen that the exponential term is directly proportional to R and inversely proportional to $\sqrt{L}$, hence it follows that a change in R makes itself felt in much greater proportion than one in L.

THE EFFECT OF LEAKANCE

Equation (14) neglects the effect of the distributed leakance G per mile. In order to obtain the operational equation for the indicial admittance when the distributed leakance G is included, replace pC by $pC + G$ in the operational equation for $h(t)$, and obtain

$$h_1(t) = \frac{pC + G}{P} . e^{-xP}$$

where $$P = \sqrt{(R+pL)(G+pC)}$$

The propagation constant P can be transformed into the form

$$P = \frac{1}{v}\sqrt{(p+\phi_1)^2 - \phi_2^2}$$

where
$$\phi_1 = \frac{R}{2L} + \frac{G}{2C}$$

$$\phi_2 = \frac{R}{2L} - \frac{G}{2C}$$

and v denotes $1/\sqrt{LC}$, as before.

The operational equation may be rewritten in the form

$$h_1(t) = \left(C + \frac{G}{p}\right)p\frac{e^{-xP}}{P}$$

$$= \frac{1}{\sqrt{LC}}\left(C + \frac{G}{p}\right)\cdot\frac{pe^{-\frac{x}{v}\sqrt{(p+\phi_1)^2-\phi_2^2}}}{\sqrt{(p+\phi_1)^2-\phi_2^2}}$$

$$\therefore\ h_1(t) = \sqrt{\frac{C}{L}}F(t) + \frac{G}{\sqrt{LC}}\int_0^t F(t)\,dt$$

which reduces the problem to an evaluation of the function $F(t)$. To evaluate this function we make use of the Carson integral solution expressed by equation (13). By putting $h_1 = \phi_1$ and $b_1 = j\phi_2$ we see that the explicit solution for $h_1(t)$ can be written as

$$h_1(t) = \sqrt{\frac{C}{L}}e^{-\phi_1 t}I_0\left(\phi_2\sqrt{t^2 - \frac{x^2}{v^2}}\right)$$

$$+ \frac{G}{\sqrt{LC}}\int_{\frac{x}{v}}^{t} e^{-\phi_1 t}I_0\left(\phi_2\sqrt{t^2 - \frac{x^2}{v^2}}\right)dt \quad . \quad . \quad (15)$$

for $$t > \frac{x}{v}$$

This equation is the exact solution for the current wave in terms of Bessel functions, and in order to evaluate the definite integral it is necessary to expand the Bessel function as a power series, integrate term by term, or else evaluate it graphically. Thus it is a matter of some labour to calculate the effect of the leakance G in any particular case.

DIRECT EXPANSION OF OPERATIONAL EQUATION

The foregoing solutions depend upon the transformation of the known Carson integral (12) into the required form of equation (13). When such identities are known their value in connexion with the solution of the operational equations which arise in electric circuit theory requires no emphasis. Of course, we should not expect to find integral solutions suitable for use with every operational equation that is likely to arise. In such cases we can always fall back on the direct Heaviside method of expanding operational equations. As an example of the direct method of expansion we will show how the function $F(t)$ can be deduced without the aid of Carson's integral equations (11) and (13).

From the formulation of $h_1(t)$ it can be seen that our problem is to evaluate the operator

$$F(t) = \frac{pe^{-\frac{x}{v}\sqrt{(p+\phi_1)^2-\phi_2^2}}}{\sqrt{(p+\phi_1)^2-\phi_2^2}}$$

Shifting the exponential $e^{-\phi_1 t}$ by means of the transformation discussed in Chapter 6, we have

$$F(t) = e^{-\phi_1 t}.\frac{p}{p-\phi_1}.\frac{(p-\phi_1)e^{-\frac{x}{v}\sqrt{(p-\phi_1+\phi_1)^2-\phi_2^2}}}{\sqrt{(p-\phi_1+\phi_1)^2-\phi_2^2}}$$

$$\therefore\ F(t) = e^{-\phi_1 t}.e^{-\frac{x}{v}\sqrt{p^2-\phi_2^2}}.\frac{p}{\sqrt{p^2-\phi_2^2}}$$

This operational equation represents a function which is zero until the time $\frac{x}{v}$. Nothing can happen at the

point x miles from the sending end until the wave arrives at x after a time interval $t = x/v$. This suggests that we transfer the time axis by means of the transfer operator $e^{p\frac{x}{v}}$. Making this transfer, the theory of which is discussed in Chapter 5, we get

$$F(t) = e^{-\phi_1 t}.e^{-p\frac{x}{v}}.e^{\frac{x}{v}(p - \sqrt{p^2-\phi_2^2})}.\frac{p}{\sqrt{p^2-\phi_2^2}}$$

We begin the development by expanding the term $\sqrt{p^2-\phi_2^2}$ by the binomial theorem; we have

$$\sqrt{p^2-\phi_2^2} = p - \frac{\phi_2^2}{2p} - \frac{\phi_2^4}{2^2 2! p^3} - \frac{3\phi_2^6}{2^3 3! p^5} - \ldots$$

Let a denote x/v, and then form the infinite product

$$e^{a(p - \sqrt{p^2-\phi_2^2})} = e^{\frac{a\phi_2^2}{2p}}.e^{\frac{a\phi_2^4}{2^2(2!)p^3}}.e^{\frac{3a\phi_2^6}{2^3(3!)p^5}} \ldots$$

Each of these expansions must be multiplied together and we then collect successive terms of inverse powers of p. Replacing p^{-n} by $t^n/n!$ in the operational expansion thus formed, we obtain

$$\begin{aligned} e^{a(p-\sqrt{p^2-\phi_2^2})} &= 1 + \frac{a\phi_2^2 t}{2} + \frac{a^2\phi_2^4 t^2}{2^2(2!)^2} \\ &\qquad + \left(\frac{a^3\phi_2^6}{2^3(3!)^2} + \frac{a\phi_2^4}{2^2(2!)(3!)}\right)t^3 + \ldots \\ &= f(t) \end{aligned}$$

Thus we may write

$$F(t) = e^{-\phi_1 t}.e^{-pa}.\frac{p}{\sqrt{p^2-\phi_2^2}}.f(t)$$

The next step is to expand the operator $p/\sqrt{p^2-\phi_2^2}$ in inverse powers of p, and substituting in the above expression for $F(t)$, we obtain

$$\begin{aligned} F(t) = e^{-\phi_1 t}.e^{-pa}\Big\{ & f(t) + \frac{1}{2}\left(\frac{\phi_2}{p}\right)^2 f(t) + \frac{1.3}{2.4}\left(\frac{\phi_2}{p}\right)^4 f(t) \\ & + \frac{1.3.5}{2.4.6}\left(\frac{\phi_2}{p}\right)^6 f(t) + \ldots\Big\} \end{aligned}$$

The next step is to perform the indicated integrations and gather the resulting series together in rising powers of t. After this the transfer operator e^{-pa} can be applied. Performing these operations, we obtain

$$F(t)=e^{-\phi_1 t}\left\{1+\left(\frac{\phi_2{}^2 a}{2}\right)(t-a)+\left(\frac{a^2\phi_2{}^4}{2^2(2!)^2}+\frac{\phi_2{}^2}{2(2!)}\right)(t-a)^2\right.$$
$$\left.+\left(\frac{a^3\phi_2{}^6}{2^3(3!)^2}+\frac{a\phi_2{}^4}{2^3(2!)^2}\right)(t-a)^3+\ldots\right\}$$

This result is due to Heaviside, who remarked that it was unfortunate that the series should be so complicated. By collecting terms in increasing powers of ϕ_2 and replacing a by x/v, Heaviside obtained the series solution

$$F(t)=e^{-\phi_1 t}\left\{1+\frac{\phi_2{}^2}{2^2}\left(t^2-\frac{x^2}{v^2}\right)+\frac{\phi_2{}^4}{2^2.4^2}\left(t^2-\frac{2x^2t^2}{v^2}+\frac{x^4}{v^4}\right)+\ldots\right\}$$

which he recognized as the power series expansion of the function $e^{-\phi_1 t}I_0\left(\phi_2\sqrt{t^2-\frac{x^2}{v^2}}\right)$. Thus the function $F(t)$ may be written from direct expansion as

$$F(t)=e^{-\phi_1 t}I_0\left(\phi_2\sqrt{t^2-\frac{x^2}{v^2}}\right)$$

which agrees with the result established by the Carson integral equation method. This example is an illustration of the fact that the direct Heaviside method of expansion may be tiresome to carry through. It must be noted, however, that only elementary mathematics are involved.

HEAVISIDE'S DISTORTIONLESS LINE

From equation (15) it can be seen that the effect of the unit voltage applied at the sending end does not reach the point x of the transmission line until a time $t=x/v$ has elapsed. Consequently $v=x/t$ is the velocity of the wave and is due to the distributed inductance L and capacity C

of the line. For if we put $R = G = 0$, then $\phi_1 = \phi_2 = 0$, and $F(t) = e^0 I_0(0) = 1$, and the indicial admittance function $h_1(t)$ reduces to

$$h_1(t) = 0 \text{ for time } t < \frac{x}{v}$$

$$= \sqrt{\frac{C}{L}} \text{ for time } t \geqslant \frac{x}{v}$$

which shows that the current wave jumps immediately to its steady-state value at time $t = x/v$. If an arbitrary electromotive force $E(t)$ is applied at the sending end at reference time $t = 0$, then the resulting current wave $i(t)$ is

$$i(t) = 0 \text{ for time } t < \frac{x}{v}$$

$$= \sqrt{\frac{C}{L}} E\left(t - \frac{x}{v}\right) \text{ for time } t \geqslant \frac{x}{v}$$

Consequently a non-dissipative transmission line transmits current waves without attenuation or distortion and with a finite velocity v. This non-dissipative transmission line is purely theoretical and quite unrealizable in practice. A distortionless line, however, may be realized physically. From a study of equation (15) Heaviside showed that the distortionless line is one in which

$$\phi_2 = \frac{R}{2L} - \frac{G}{2C} = 0$$

and that $$h(t) = 0 \text{ for time } t < \frac{x}{v}$$

$$= \sqrt{\frac{C}{L}} e^{-\beta x} \text{ for time } t \geqslant \frac{x}{v}$$

where $$\beta = \frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}}$$

This shows that the current wave is transmitted without distortion of wave form. If an electromotive-force $E(t)$ is

applied at the sending end, then the resulting current wave $i(t)$ is

$$i(t) = 0 \text{ for time } t < \frac{x}{v}$$

$$= \sqrt{\frac{C}{L}}e^{-\beta x}E\left(t - \frac{x}{v}\right) \text{ for time } t \geqslant \frac{x}{v}$$

These equations show that the distortionless line transmits the current waves without distortion of wave form, but attenuates the waves by the factor $e^{-\beta x}$. Thus the condition $\phi_2 = 0$ preserves the wave shape but introduces serious attenuation losses. It was one of Heaviside's great achievements to deduce the properties of the distortionless transmission line and to point out its approximately realizable character and to base thereon an engineering theory of telephonic transmission.

HEAVISIDE'S DIVERGENT EXPANSIONS

The series obtained for results so far have all been convergent. It was shown in Chapter 4 that these series are convenient for calculation only when the time t is small. When t is large, many terms need to be taken into account. Heaviside found that it is often possible to obtain easily an alternative form of series, which, in spite of the fact that it is divergent, can be used for computation when t is large. In order to fix ideas, let us consider an example; in Chapter 4 we discussed the Bessel function $e^{-\phi t}I_0(\phi t)$ and found that it was a simple matter to obtain a convergent power series expansion in the form

$$e^{-\phi t}I_0(\phi t) = 1 - (\phi t) + \frac{1\cdot 3}{(2!)^2}(\phi t)^2 - \frac{1\cdot 3\cdot 5}{(3!)^2}(\phi t)^3 + \dots$$

This convergent solution is convenient for calculation when t is small. When t is large, however, this solution is difficult to use owing to its slow convergence. In order to

overcome this difficulty it is possible to obtain a divergent and asymptotic expansion of the form

$$e^{-\phi t}I_0(\phi t) = \frac{1}{\sqrt{2\pi\phi t}}\left\{1 + \frac{1}{(8\phi t)} + \frac{1^2.3^2}{2!(8\phi t)^2} + \frac{1^3.3^3.5^3}{3!(8\phi t)^3} + \ldots\right\}$$

Now for relatively large values of t the successive terms at first become smaller, and as we proceed along the series we reach a smallest term. After this, however, the terms begin to increase and ultimately become very large. These characteristics are illustrated in Fig. 10.

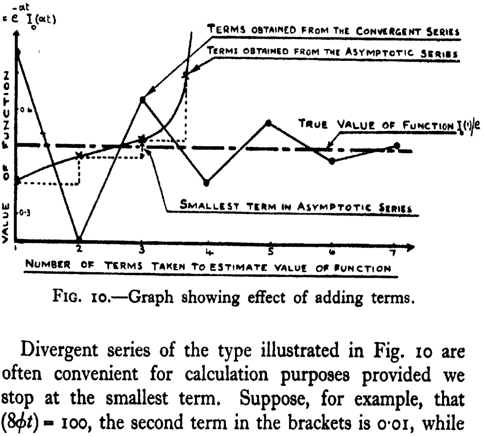

FIG. 10.—Graph showing effect of adding terms.

Divergent series of the type illustrated in Fig. 10 are often convenient for calculation purposes provided we stop at the smallest term. Suppose, for example, that $(8\phi t) = 100$, the second term in the brackets is 0·01, while the third term is only 0·00045. The larger the value of $(8\phi t)$ the fewer the terms that need be used. For example, if $(8\phi t) > 100$ the series reduces to $1/\sqrt{2\pi\phi t}$. In using such series for calculating the results of electric circuit problems we note that we must stop at the smallest term, beyond which it is useless to proceed.

In order to show how these series arise in electric circuit

theory, put $x = 0$ in the operational expression for $h(t)$ and obtain

$$h_0(t) = \sqrt{\frac{pC}{R + pL}}$$

for the current entering the transmission line in response to a one-volt battery. There are several ways of solving this operational equation. Perhaps the quickest way is to search for the required form in a table of Carson integral solutions. Writing the expression for $h_0(t)$ as $\sqrt{\frac{C}{L}} \cdot \frac{1}{\sqrt{1 + \frac{R}{pL}}}$, we note that the solution (E) in the short table of Carson integrals given in Chapter 6 gives the required form, namely,

$$\frac{1}{\sqrt{1 + \frac{2\phi}{\lambda}}} = \lambda \int_0^\infty e^{-\lambda t} . e^{-\phi t} I_0(\phi t)\, dt$$

provided we put $2\phi = R/L$. From this result we see that the required solution is

$$h_0(t) = \sqrt{\frac{C}{L}} e^{-\phi t} I_0(\phi t) \quad . \quad . \quad . \quad . \quad (16)$$

where $e^{-\phi t} I_0(\phi t)$ is a tabulated Bessel function.

The equation (16) may be readily established by other methods. We may write

$$h_0(t) = \sqrt{\frac{C}{L}} \cdot \frac{1}{\sqrt{1 + \frac{2\phi}{p}}}$$

where $\phi = R/2L$, and expanding by the binomial theorem, we obtain

$$h_0(t) = \sqrt{\frac{C}{L}} \left\{ 1 - \left(\frac{\phi}{p}\right) + \frac{1 \cdot 3}{2!}\left(\frac{\phi}{p}\right)^2 - \frac{1 \cdot 3 \cdot 5}{3!}\left(\frac{\phi}{p}\right)^3 + \ldots \right\}$$

In order to convert this operational expansion into an explicit power series solution, we replace p^{-n} by $t^n/n!$ in

accordance with the principles discussed in Chapter 3. The explicit solution is then

$$h_0(t) = \sqrt{\frac{C}{L}}\left\{1 - (\phi t) + \frac{1\cdot 3}{(2!)^2}(\phi t)^2 - \frac{1\cdot 3\cdot 5}{(3!)^2}(\phi t)^3 + \ldots\right\}$$

which is a convergent series solution of the operational equation in rising powers of t. This series solution is convenient for computing $h_0(t)$ when t is small, i.e. for calculating the first transient surge in electric circuit problems. When t is large, however, this series is difficult to use owing to its slow convergence. In order to overcome this difficulty it is necessary to recognize and sum the series. After a trial it will be found that the series is the convergent power series expansion of the tabulated Bessel function in equation (16). If we are unable to recognize and sum the power series expansion and wish to calculate $h_0(t)$ for large values of t, it will be necessary to use the Heaviside scheme of expanding the operator in rising powers of p, from which a convenient series for calculating purposes can be obtained. The operational equation for $h_0(t)$ may be written in the form

$$h_0(t) = \sqrt{\frac{C}{L}}\cdot\frac{\sqrt{\dfrac{p}{2\phi}}}{\sqrt{1+\dfrac{p}{2\phi}}}$$

Expanding the denominator by the binomial theorem, we get

$$h_0(t) = \sqrt{\frac{C}{L}}\left\{1 - \left(\frac{p}{4\phi}\right) + \frac{1\cdot 3}{2!}\left(\frac{p}{4\phi}\right)^2 - \frac{1\cdot 3\cdot 5}{3!}\left(\frac{p}{4\phi}\right)^3 + \ldots\right\}\sqrt{\frac{p}{2\phi}}$$

It will be observed that the operand is now $\sqrt{p}[1]$, and consequently we make use of the fact, rigorously established in Chapter 6, that $\sqrt{p}[1]$ must be interpreted as $1/\sqrt{\pi t}$.

Making this substitution and replacing p^n by $\frac{d^n}{dt^n}$ we obtain

$$h_0(t) = \sqrt{\frac{C}{L}} \cdot \left\{ 1 - \frac{1}{4\phi} \cdot \frac{d}{dt} + \frac{1 \cdot 3}{2!} \cdot \frac{1}{(4\phi)^2} \cdot \frac{d^2}{dt^2} - \frac{1 \cdot 3 \cdot 5}{3!} \cdot \frac{1}{(4\phi)^3} \cdot \frac{d^3}{dt^3} + \ldots \right\} \frac{1}{\sqrt{2\phi}} \cdot \frac{1}{\sqrt{\pi t}}$$

Performing the indicated differentiations, we obtain

$$h_0(t) = \sqrt{\frac{C}{L}} \cdot \frac{1}{\sqrt{2\pi\phi t}} \cdot \left\{ 1 + \frac{1}{(8\phi t)} + \frac{(1 \cdot 3)^2}{2!(8\phi t)^2} + \frac{(1 \cdot 3 \cdot 5)^3}{3!(8\phi t)^3} + \ldots \right\}$$

This solution may be used to compute $h_0(t)$ when t is large, and is, in fact, the asymptotic expansion of the function $e^{-\phi t} I_0(\phi t)$. The asymptotic expansion of this function is usually derived by intricate and difficult mathematical processes and the remarkable ease of the Heaviside method should be noted.

CABLE WITH TERMINAL RESISTANCE

As an example of Heaviside's method of obtaining two expansions (one for small values of t and the other for large values of t), consider the problem of the determination of the voltage $e_0(t)$ which appears at the terminals of a non-inductive cable when a one-volt battery is applied through a terminal resistance R_0. The arrangement is shown in Fig. 11.

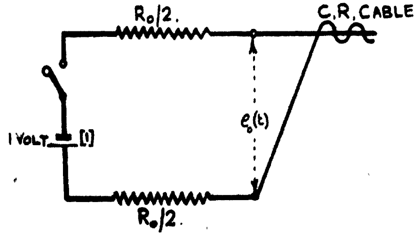

FIG. 11.—Non-inductive cable with terminal resistance.

The impedance function of the resistance R_0 and the cable in series, is

$$Z(p) = R_0 + \sqrt{\frac{R}{pC}}$$

where R = resistance per mile of cable,
C = capacity per mile of cable.
The entering current $h_0(t)$ is given by

$$h_0(t) = \frac{1}{R_0 + \sqrt{R/pC}} \quad . \quad . \quad . \quad . \quad (17)$$

and the voltage $e_0(t)$ at the cable terminals is

$$e_0(t) = 1 - \frac{R_0}{R_0 + \sqrt{R/pC}} \quad . \quad . \quad . \quad . \quad (18)$$

In order to obtain a convergent power series solution we must expand the operational equation (17) in inverse powers of p. Writing equation (17) in the form

$$h_0(t) = \frac{1}{R_0} \cdot \frac{1}{1 + 1/(hp)^{\frac{1}{2}}}$$

where $$\sqrt{h} = R_0\sqrt{\frac{C}{R}}$$

We obtain on expanding and collecting terms

$$h_0(t) = \frac{1}{R_0}\left\{1 + \frac{1}{(hp)} + \frac{1}{(hp)^2} + \ldots\right\} - \frac{(hp)^{\frac{1}{2}}}{R_0}\left\{\frac{1}{(hp)} + \frac{1}{(hp)^2} + \ldots\right\}$$

Replacing p^{-n} by $t^n/n!$ and $\sqrt{p}$ by $1/\sqrt{\pi t}$, we obtain

$$h_0(t) = \frac{1}{R_0}\left\{1 + \left(\frac{t}{h}\right) + \frac{1}{2!}\left(\frac{t}{h}\right)^2 + \ldots\right\}$$

$$- \frac{\sqrt{h}}{R_0\sqrt{\pi}}\left\{\frac{1}{(hp)}t^{-\frac{1}{2}} + \frac{1}{(hp)^2}t^{-\frac{1}{2}} + \ldots\right\}$$

Performing the indicated integrations in the second series

and recognizing the first series as the expansion of the exponential $e^{t/h}$, we obtain

$$h_0(t) = \frac{e^{\frac{t}{h}}}{R_0} - \frac{2}{R_0}\sqrt{\frac{t}{\pi h}}\left\{1 + \frac{2}{3}\left(\frac{t}{h}\right) + \frac{2^2}{3\cdot 5}\left(\frac{t}{h}\right)^2 + \ldots\right\}$$

Substituting in equation (18) we obtain for the terminal voltage $e_0(t)$ the expression

$$e_0(t) = 1 - e^{\frac{t}{h}} + \frac{2}{\sqrt{\pi}}\sqrt{\frac{t}{h}}\left\{1 + \frac{2}{3}\left(\frac{t}{h}\right) + \frac{2^2}{3\cdot 5}\left(\frac{t}{h}\right)^2 + \ldots\right\}$$

This result is convergent and convenient for calculation when t is small. Although this series is valid for any value of t it is hard to compute from when t is large. It will be noted that the voltage $e_0(t)$ contains a positive exponential term; this voltage, however, does not increase exponentially, because for large values of t, the exponential term is practically cancelled by the series that follows it. In order to obtain an asymptotic expansion suitable for calculation when t is large, we must expand the operational equation (17) in ascending powers of p. From equation (17) we have, after a simple rearrangement

$$h_0(t) = \frac{1}{R_0}\cdot\frac{(hp)^{\frac{1}{2}}}{1 + (hp)^{\frac{1}{2}}} \quad \text{where } \sqrt{h} = R_0\sqrt{\frac{C}{R}}$$

We obtain on expansion

$$h_0(t) = \frac{1}{R_0}(hp)^{\frac{1}{2}}\{1 - (hp)^{\frac{1}{2}} + (hp) - (hp)^{\frac{3}{2}} + (hp)^2 - \ldots\}$$

Now for the finite values of the time we are concerned with, $p^n[1] = 0$, where n is any positive integer. Thus the integral powers of p may be dropped, and rearranging we have

$$h_0(t) = \frac{(hp)^{\frac{1}{2}}}{R_0}\{1 + (hp) + (hp)^2 + (hp)^3 + \ldots\}$$

Replacing $\sqrt{p}$ by $1/\sqrt{\pi t}$, and performing the indicated differentiations, we obtain

$$h_0(t) = \frac{1}{R_0\sqrt{\pi}}\left(\frac{h}{t}\right)^{\frac{1}{2}}\left\{1 - \frac{1}{2}\left(\frac{h}{t}\right) + \frac{3}{2^2}\left(\frac{h}{t}\right)^2 - \ldots\right\}$$

Thus, the voltage $e_0(t)$ across the cable terminals is

$$e_0(t) = 1 - \frac{1}{\sqrt{\pi}}\sqrt{\frac{h}{t}}\cdot\left\{1 - \frac{1}{2}\left(\frac{h}{t}\right) + \frac{3}{2^2}\left(\frac{h}{t}\right)^2 - \ldots\right\}$$

which is an asymptotic expansion convenient for computation when t is large. It must be noted that the error involved in using an asymptotic series does not become negligible until t becomes large. This, of course, is all that can be expected from an asymptotic series; if t is small it is a simple matter to use the ordinary convergent series.

This example is an illustration of Heaviside's method of obtaining divergent and asymptotic series solutions of electric circuit problems. Heaviside found that if an operational equation, $h(t) = 1/Z(p)$, admits of series expansion in the form

$$h(t) = C_0 + C_1\sqrt{p} + C_2 p + C_3 p\sqrt{p} + C_4 p^2 + \ldots$$

a solution, usually divergent and asymptotic, results from discarding the terms in integral powers of p, and replacing $p^n\sqrt{p}$ by $\frac{d^n}{dt^n}\cdot\frac{1}{\sqrt{\pi t}}$. Thus the explicit solution may be written

$$h(t) = C_0 + \left\{C_1 + C_3\frac{d}{dt} + C_5\frac{d^2}{dt^2} + \ldots\right\}\frac{1}{\sqrt{\pi t}}$$

where the C's are constants.

It can be seen that Heaviside's asymptotic expansions may be of great value in certain kinds of electric circuit investigations. The question of when and how divergent and asymptotic series may be rigorously developed involves advanced mathematical considerations. Heaviside did not

attempt to explain the mathematical theory; the only justification he gave was an appeal to direct measurement to check his calculations. He appeared to have evolved a simple working system based on trial, confirmed on test, and then applied to electric circuit problems on a large scale. By virtue of his last theorem, however, it would have been an easy matter for Heaviside to have investigated the mathematical validity of his operational methods. Thus if he split his operator $F(p)$ into a sum of terms and if when he evaluated each term by his integral theorem they added up to $F(\lambda)/\lambda$, then the operational process would be justified: there are, of course, other methods. It is possible that Heaviside made these investigations but never bothered to publish them; in this case they will probably be found reposing peacefully in the archives of the Institution of Electrical Engineers together with other unpublished work of Heaviside. Since the time of Heaviside, however, the theory of his divergent and asymptotic expansions has been placed on a rigorous mathematical footing by Wiener and Carson.

CHAPTER 8

THE APPLICATION OF MODERN THEORIES OF INTEGRATION TO THE SOLUTION OF CIRCUIT PROBLEMS

COMPLEX INTEGRATION

IN Chapter 5 it was stated that the Heaviside operational equation is not a true algebraic equation at all but is simply a shorthand way of writing down the descriptive differential equations of the network. It was also shown that to every operational equation there corresponds a Carson integral equation. This integral equation expresses a unique but implicit relationship between $h(t)$ and $Z(\lambda)$, which enables us to extend the range of operational solutions by direct transformations. The formal solution of Carson's integral equation is given by the Bromwich contour integral which gives an explicit relationship between $h(t)$ and $Z(\lambda)$. By virtue of the Carson-Bromwich relationships it is possible to establish rigorously all the fundamental theorems of the Heaviside operational calculus.

It may seem curious to the student that the operational calculus, invented by Heaviside for dealing with functions of a real variable, should find it necessary to use the theory of functions of a complex variable in order to establish rigorously its fundamental theorems. This is because the descriptive differential equations which arise in network problems are expressible in terms of analytic functions. These analytic functions are found to satisfy Cauchy's integral theorems. Consequently use is made of the calculus of residues in order to evaluate the infinite integrals which arise.

In the theory of functions of a real variable we consider functions associated with the movement of points along a straight line, and begin the study of these functions by

considering the class of functions which possess derivatives of all orders and are capable of expansion in a power series by Taylor's theorem about any point on the line. In the theory of functions of a complex variable, however, we are dealing with functions associated with movements of points in a plane, and begin the study of these functions by considering the restricted class of analytic functions which possess derivatives of all orders and are capable of expansion in a power series by Taylor's theorem about any point in a given region in the plane. Thus if $f(z)$ is analytic in a given region in the complex plane it can be expanded in a Taylor series about any point $z = a$ in that region and

$$f(z) = \sum_{0}^{\infty} a_n (z - a)^n$$

In order to apply complex variable theory to electric circuit problems we begin by considering the meaning of the integral of a function of a complex variable along a plane curve. Now the physical ideas associated with line integrals in the complex plane are similar to those associated with integrals taken along the real axis, except that the integrand is of the form $x + jy$, and consequently the differential is also complex. For example, the meaning of $\int f(z)\, dz$ between any two points in the complex plane may be visualized as follows: the path of integration may be divided into tiny segments dz, and this elemental length is multiplied by the value of the function $f(z)$ at the midpoint of the segment. We then add up all these elemental products and take the limit of the sum as $\delta z \rightarrow 0$. This limiting sum is then the value of the integral $\int f(z)\, dz$ between the two points considered. It follows from these considerations that the integral around a closed path is zero, provided there are no singularities or points of discontinuity on or within the path of integration. This statement concerning the value of $f(z)$ taken round a closed

contour is the basis of a fundamental theorem due to Cauchy, which may be stated formally as follows: if $f(z)$ is analytic at all points on and inside a closed contour C, then $\int f(z)\,dz = 0$. Because of the fact that the line integral taken a closed contour C is zero, it follows that the path of integration may be stretched or contracted in any way we please, provided in so doing we do not pass over any singularities of the function, that is, we must not pass over points at which $f(z)$ becomes infinite. Should the area contained within the path of integration enclose a singularity, then we may contract the path until parts coincide and cancel, and we are left with only a small circle surrounding the singularity. These ideas are illustrated in Fig. 12,

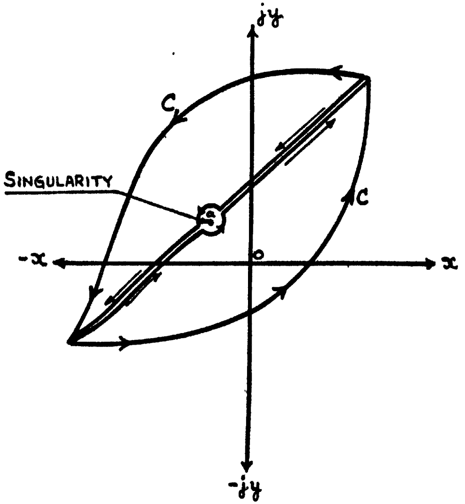

FIG. 12.—Path of integration in the complex plane.

where C is a contour which contains a singularity or point of discontinuity a.

It can be seen from Fig. 12 that if the contour C is contracted so that the parts coincide and cancel, we are left with only a small circular contour surrounding the singularity at a.

These ideas are important in electric circuit theory; for example, in the examination of a circuit problem by means of Heaviside's expansion theorem we see that the roots of the equation, $Z(\lambda) = 0$, play an important part. For dissipative systems these roots are complex with negative real parts, and, if plotted in the complex plane, they will generally lie on the left-hand side of the axis of imaginaries. If, however, the system contains negative resistances, then it is possible that the complex roots may have positive real parts. These roots are related to the singularities of the function $1/Z(\lambda)$ and determine the nature of the operator with which we have to deal and the process which may be safely employed to interpret it.

If $f(z)$ is analytic in a given domain in the complex plane, except at the singularity $z = a$, then we can draw two concentric circles of centre a, both lying within the domain. In the annulus between these two circles, we have by virtue of Laurent's theorem, an expansion of the form

$$f(z) = \sum_{0}^{\infty} a_n(z-a)^n + \sum_{1}^{\infty} b_n(z-a)^{-n}$$

The second term on the right-hand side is called the 'principal part' of $f(z)$ at the singularity $z = a$. It may happen that $b_{n+1} = b_{n+2} = b_{n+3} = \ldots = 0$ but $b_n \neq 0$. In this case the principal part consists of the finite number of terms

$$\frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_n}{(z-a)^n}$$

and the singularity at $z = a$ is called 'a pole of order n' of $f(z)$ and the coefficient b_1 of $1/(z-a)$ is called 'the residue of $f(z)$ at the pole $z = a$'. If the principal part is

an infinite series the singularity at $z = a$ is called 'an essential singularity'. In electric circuit theory the functional singularities are usually first-order poles or essential singularities. From the above definition it can be seen that b_1, the residue of the function at a first-order pole, is given by $b_1 = \lim_{z \to a} (z - a)f(z)$.

It follows from Cauchy's theorem that the value of a line integral about a closed contour is equal to the sum of the integrals about small circles in the original area each enclosing a pole. It is an easy matter to show that the line integral of a function taken around one of these small circles is equal to $2\pi j$ times the residue b_1 of the function at the pole. For example, suppose we integrate the simple function $1/\lambda$ about one of these circles. It will be noted that this function has a pole at the origin and consequently the line integral of $1/\lambda$ about the origin is $2\pi j$, for the residue there is $\lim_{\lambda \to 0} \lambda\left(\frac{1}{\lambda}\right)$ which is unity. If, however, the circle did not enclose the pole at the origin, the value of the line integral of $1/\lambda$ would be zero. This can be easily seen by integrating round the circle shown in Fig. 13.

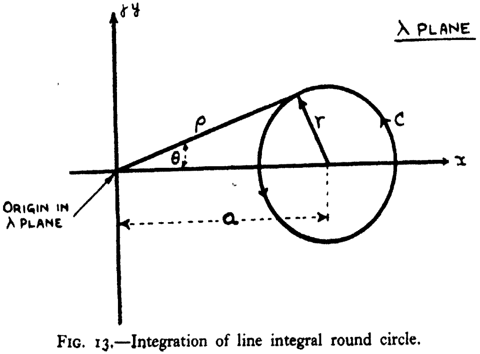

FIG. 13.—Integration of line integral round circle.

Putting $\lambda = \rho e^{j\theta}$, we have

$$\int_C \frac{d\lambda}{\lambda} = [\log \lambda]_C = [\log |\lambda| + j\theta]_C$$

$$\therefore \int_C \frac{d\lambda}{\lambda} = j \text{ (change in } \theta)$$

It can be seen that the change in θ is o if $r < a$ and 2π if $r > a$, which is in agreement with the result established by the residue method. The statement that the value of a line integral round a small circle enclosing a pole, is equal to $2\pi j$ times the residue b_1, of the function at the pole, is a special case of Cauchy's Residue Theorem which may be stated formally as follows: if $f(z)$ be analytic on and within a closed contour C, except for a finite number of poles within C, then

$$\int_C f(z)\, dz = 2\pi j \Sigma R \qquad .$$

where ΣR is the sum of the residues of $f(z)$ at its poles within C. In the majority of electric circuit problems the poles which arise are mostly first-order poles and consequently the residue at a pole of $f(z)$ may be calculated from $\lim_{z \to a} (z - a)f(z)$.

The application of Cauchy's residue theorem to electric circuit theory is important because it will enable a student to see clearly the functional aspects of Heaviside's operational calculus. It will enable him to extend the method to unusual problems and to see danger signals in doubtful cases. He will also find that an argument in terms of integrals in the complex plane is very much simpler than one in terms of ordinary integrals.

THE BROMWICH CONTOUR INTEGRAL

We have yet to demonstrate that the Bromwich contour integral (9) is the formal solution of the Carson integral

equation (8). There are several ways of doing this; we recall that the function $h(t)$ is uniquely but implicitly determined by the Carson relationship, and consequently the problem is to formulate $h(t)$ as an explicit function of $Z(\lambda)$. Since we are primarily interested in the application of operators to a discontinuous operand, we begin by formulating the unit function [1] as a contour integral. This unit function is zero before, unity after time $t = 0$, and is consequently discontinuous at the origin. From Cauchy's residue theorem it is apparent that the unit function can be expressed as

$$[1] = \frac{1}{2\pi j}\int_C f(\lambda)\, d\lambda \quad . \quad . \quad . \quad . \quad . \quad (19)$$

provided we choose a suitable contour and a function of λ that is everywhere regular in the λ plane, except at the origin, where it must be discontinuous and have a residue of unity. It is a simple matter to see that such a function is $e^{\lambda t}/\lambda$. This function has a pole at the origin in the λ plane and its residue there is $\lim\limits_{\lambda \to 0} \lambda\left(\frac{e^{\lambda t}}{\lambda}\right)$, which is unity. Consequently $f(\lambda)$ in equation (19) may be taken as $e^{\lambda t}/\lambda$, and C may be a circle surrounding the origin in the λ plane as shown in Fig. 14.

We know that it is the reciprocal of the impedance function that acts on the unit function and that the complex roots of $Z(\lambda) = 0$ have negative real parts for all dissipative systems. We must therefore see that all the singularities of the integrand are on the left-hand side of our contour. Consequently we enclose the origin in the λ plane by a small half-circle so as to include it in the left-hand semicircle as shown in the upper part of Fig. 14.

Now the value of the line integral taken round the right-hand contour $ABDA$ in the upper part of Fig. 14 is obviously zero because the path of integration is regular and the enclosed area contains no poles. The value of

the line integral taken round the left-hand contour $ADEA$, however, is $2\pi j$ times the residue at the origin, which in this case is unity. Consequently it is possible to write equation (19) in the form

$$[1] = \frac{1}{2\pi j}\int_{C'} \frac{e^{\lambda t}}{\lambda}\, d\lambda \quad . \quad . \quad . \quad . \quad . \quad (20)$$

with the contour shown in Fig. 14. An equivalent path

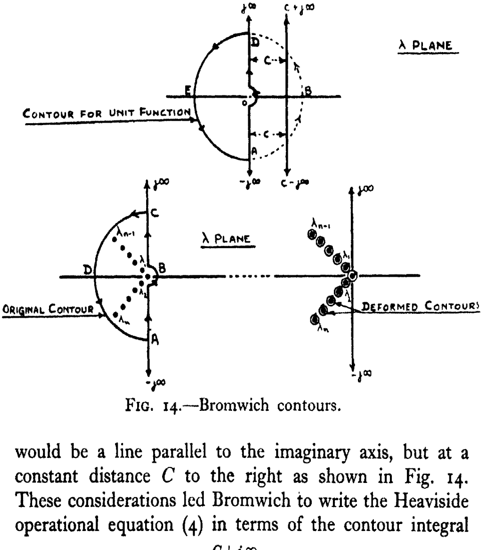

FIG. 14.—Bromwich contours.

would be a line parallel to the imaginary axis, but at a constant distance C to the right as shown in Fig. 14. These considerations led Bromwich to write the Heaviside operational equation (4) in terms of the contour integral

$$h(t) = \frac{1}{2\pi j}\int_{C-j\infty}^{C+j\infty} \frac{e^{\lambda t}}{\lambda Z(\lambda)}\, d\lambda \quad . \quad . \quad . \quad . \quad (9)$$

where C is so chosen that all the singularities of the integrand lie on the left of the path of integration in the

complex plane. In the majority of electric circuit problems the constant C may be taken as zero; there are exceptions, however, and cases may arise when the Bromwich equation (9) is valid only when C is made greater than some finite constant. Such an exceptional case arises when the electrical system specified by $Z(\lambda)$ is 'unstable', that is, it may contain some internal source of energy or 'negative' resistance which makes its transient oscillations increase with time. In such a case, $Z(\lambda)=0$ will have roots to the right of the imaginary axis, and in order that the Bromwich contour integral shall always give the correct interpretation of its corresponding operational equation, C must be large enough to ensure that all the singularities of the integrand lie on the left of the path of integration.

DERIVATION OF EXPANSION THEOREMS

In Chapter 2 we showed that the function $1/Z(\lambda)$ can be expressed in terms of partial fractions, and that

$$\frac{1}{Z(\lambda)}=\sum_{K=1}^{n}\frac{1}{(\lambda-\lambda_K)Z'(\lambda_K)}$$

Substituting this result in the Bromwich equation (9), we obtain

$$h(t)=\frac{1}{2\pi j}\int_{C-j\infty}^{C+j\infty}e^{\lambda t}\sum_{K=1}^{n}\frac{1}{\lambda(\lambda-\lambda_K)Z'(\lambda_K)}d\lambda \quad . \quad (21)$$

In order to apply Cauchy's residue theorem to the evaluation of this integral, we first note that the integrand has poles at $0, \lambda_1, \lambda_2, \ldots \lambda_K, \ldots \lambda_n$. Some of these poles and the surrounding contours are shown graphically in Fig. 14. The contour on the left-hand side includes within its area the poles at $0, \lambda_1, \lambda_2, \ldots \lambda_n$. We may now contract the contour $ABCDA$ in such a way that we are left with only small circles surrounding the poles as shown on the right-hand side of Fig. 14. We must now evaluate

the residues at the poles of the integrand in equation (21). The residues at the pole at the origin of the integrand is

$$R_0 = \lim_{\lambda \to 0} \lambda . \frac{e^{\lambda t}}{\lambda Z(\lambda)} = \frac{1}{Z(0)}$$

which is the steady-state term.

The residue at the pole λ_K of the integrand is

$$R_K = \lim_{\lambda \to \lambda_K} (\lambda - \lambda_K) . \frac{e^{\lambda t}}{\lambda(\lambda - \lambda_K)Z'(\lambda_K)}$$

$$\therefore\ R_K = \frac{e^{\lambda_K t}}{\lambda_K Z'(\lambda_K)} \quad (K = 1,\ 2,\ 3,\ \ldots\ n)$$

Hence the sum of the residues is

$$\Sigma R = \frac{1}{Z(0)} + \sum_{K=1}^{n} \frac{e^{\lambda_K t}}{\lambda_K Z'(\lambda_K)}$$

and by virtue of the Bromwich formulation we may write

$$h(t) = \frac{1}{Z(0)} + \sum_{K=1}^{n} \frac{e^{\lambda_K t}}{\lambda_K Z'(\lambda_K)}$$

which is the well-known Heaviside expansion theorem. This theorem has now been deduced by a rigorous mathematical method and verifies the simple experimental method used in Chapter 2.

The partial fraction expansion used in this derivation shows that the roots of the impedance function, $Z(\lambda) = 0$, must be unequal and distinct from each other. In all finite dissipative networks this condition is satisfied. The case of repeated roots can be dealt with by the method used in Chapter 2, namely, letting the roots approach equality as a limit.

This method of derivation may be easily extended to cover alternating voltages; suppose, for example, that instead of the unit voltage [1], we applied the alternating voltage $E \cos \omega t[1]$ to the network at time $t = 0$. Let it be required to determine the current response $i(t)$ of a network described by the impedance function $Z(\lambda)$. For

purposes of analysis we may write this alternating voltage as the real part of $Ee^{j\omega t}[1]$; and from the Bromwich equation (20) it can be seen that this type of voltage may be represented by the contour integral

$$Ee^{j\omega t}[1] = \frac{E}{2\pi j}\int_C \frac{e^{(\lambda+j\omega)t}}{\lambda}.d\lambda$$

with the same contour as for equation (20). It follows from our discussion of the differential equations of circuit theory that if the applied voltage is of the form, $e^{(\lambda+j\omega)t}$, then $Z(\lambda+j\omega)$ will represent the impedance function of the network. Thus we may write $i(t)$ as the Bromwich contour integral

$$i(t) = \frac{E}{2\pi j}\int_{C-j\infty}^{C+j\infty} \frac{e^{(\lambda+j\omega)t}}{\lambda Z(\lambda+j\omega)}\,d\lambda \quad . \quad . \quad . \quad (22)$$

The integrand has poles at 0, $(\lambda_1 - j\omega)$, $(\lambda_2 - j\omega)$, ... $(\lambda_K - j\omega)$, ... $(\lambda_n - j\omega)$.
The residue R_0 at the origin is

$$R_0 = \lim_{\lambda\to 0} \lambda \frac{Ee^{(\lambda+j\omega)t}}{\lambda Z(\lambda+j\omega)} = \frac{Ee^{j\omega t}}{Z(j\omega)}$$

which is the steady-state term.
The residue R_K at the pole $(\lambda_K - j\omega)$ is

$$R_K = \lim_{\lambda\to(\lambda_K - j\omega)} \{\lambda - (\lambda_K - j\omega)\}.\frac{Ee^{(\lambda+j\omega)t}}{\lambda\{\lambda - (\lambda_K - j\omega)\}Z'(\lambda+j\omega)}$$

$$\therefore\ R_K = \frac{Ee^{\lambda_K t}}{(\lambda_K - j\omega)Z'(\lambda_K)} \qquad (K = 1,\ 2,\ 3,\ \ldots\ n)$$

Hence $i(t)$, which is the sum of the residues, may be written

$$i(t) = \frac{Ee^{j\omega t}}{Z(j\omega)} + E\sum_{K=1}^{n}\frac{e^{\lambda_K t}}{(\lambda_K - j\omega)Z'(\lambda_K)}$$

which is the alternating-current expansion theorem discussed in Chapter 3. The first term on the right-hand side of the alternating-current expansion theorem represents

the steady-state current, while the summation terms give the transient oscillations. The alternating-current expansion theorem was given by Carson; it will be noted that when $\omega = 0$ and $E = 1$, Carson's expansion theorem reduces to Heaviside's expansion theorem as it should do.

A RELAY PROBLEM

The Carson expansion theorem is sometimes very useful for the solution of alternating-current transient problems which arise in connexion with the design of lumped circuits. As an example, suppose it is required to investigate the transient oscillations in the circuit shown in Fig. 15.

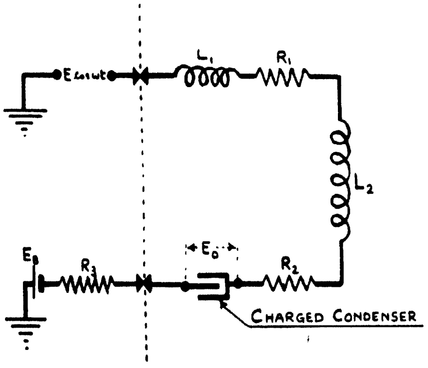

FIG. 15.—Relay circuit.

At reference time $t = 0$, a direct voltage E_B and an alternating voltage $E \cos \omega t$ are applied to a circuit containing resistive and inductive elements as well as a charged condenser. It is required to investigate the transient oscillations in this circuit, which is part of a relay arrangement used in automatic telephony.

From the principle of superposition we know that at time $t \geqslant 0$, the current $i(t)$ in the circuit will be

$$i(t) = i_1(t) + i_2(t) + i_3(t)$$

where $i_1(t)$ = current due to discharge of condenser
$i_2(t)$ = current due to direct voltage E_B
$i_3(t)$ = current due to alternating voltage.

In order to obtain an expression for $i_1(t)$ we note that if a battery of voltage E_0 be suddenly connected in series with an uncharged condenser, the resulting current is the same as when the condenser is initially charged to a voltage E_0 and then discharged through the circuit, the direction of the voltage being maintained the same in the two cases. This follows because in each case the voltage $e(t)$ impressed on the circuit is initially E_0 and is decreased by an amount, $\frac{1}{C}\int i\,dt$, as the current flows. Since the voltage impressed on the circuit is the same in each instant, the same current will flow, and one may be analysed in place of the other. Consequently the problem of an initially charged condenser is reduced to that of a direct voltage applied to an uncharged condenser. Thus the operational equation for $i_1(t)$ may be written as

$$i_1(t) = \frac{E_0}{R + pL + 1/pC}$$

where

$$R = R_1 + R_2 + R_3$$
$$L = L_1 + L_2.$$

Also E_0 may be replaced by Q/C, where Q is the initial charge on the condenser in coulombs and C is the capacity in farads. It will be recalled that the solution of this operational equation was obtained in Chapter 2. We are here interested in the case where $\frac{1}{L} > \frac{R^2}{4L^2}$; thus we may write

$$i_1(t) = \frac{Q}{\beta LC} \cdot e^{-\alpha t} \sin \beta t$$

where $\alpha = R/2L$ and $\beta = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$

The operational equation for $i_2(t)$ is the same as for $i_1(t)$, except that E_0 is now replaced by E_B, the direct battery voltage. Thus we may write

$$i_2(t) = \frac{E_B}{\beta L} e^{-\alpha t} \sin \beta t$$

An expression for $i_3(t)$ can be obtained by applying the Carson expansion theorem. The values of λ_1, λ_2 and $\partial Z(\lambda)/\partial\lambda$ are, of course, the same as those obtained in Chapter 2 for the impedance function $Z(p) = R + pL + 1/pC$. Substituting for $i_3(t)$ we obtain

$$i_3(t) = \frac{Ee^{j\omega t}}{R+j\omega L+1/j\omega C} - \frac{Ee^{-\alpha t}}{2\beta L}\left\{\frac{(\beta+j\alpha)e^{j\beta t}}{j\omega+\alpha-j\beta} + \frac{(\beta-j\alpha)e^{-j\beta t}}{j\omega+\alpha+j\beta}\right\}$$

Simplifying this expression and then taking the real part, we obtain

$$i_3(t) = \frac{E\cos(\omega t - \phi)}{\sqrt{R^2 + X^2}} + \frac{Ee^{-\alpha t}}{\beta L} \cdot \frac{(\alpha^2 + \beta^2)\sin(\beta t - \psi)}{\sqrt{(\alpha^2 + \beta^2 - \omega^2)^2 + 4\alpha^2\omega^2}}$$

where
$$\tan\phi = X/R$$
$$X = \omega L - 1/\omega C$$
$$\tan\psi = 2\omega^2\alpha\beta/\{(\alpha^2 + \beta^2)^2 + \omega^2(\alpha^2 - \beta^2)\}$$

For $\omega = 0$, the steady-state component becomes zero and the alternating-current transient component reduces to the direct-current transient. If the transient distortion D be defined as the instantaneous difference between the actual response $i(t)$ of the relay circuit and the steady-state response, we see that the transient distortion may be written as

$$D = \frac{e^{-\alpha t}}{\beta L}\left\{\left(\frac{Q}{C} + E_B\right)\sin\beta t + \frac{E(\alpha^2 + \beta^2)\sin(\beta t - \psi)}{\sqrt{(\alpha^2 + \beta^2 - \omega^2)^2 + 4\alpha^2\omega^2}}\right\}$$

Now Q/C is the initial voltage E_0 across the condenser at reference time $t = 0$, and it can be seen that if the battery voltage E_B is made equal and opposite to E_0, the first term

on the right-hand side of this equation will be zero. This shows that the transient distortion in the relay circuit may be considerably reduced by making E_B oppose E_0. For low frequencies the second term on the right-hand side of the equation is small.

THE SUPERPOSITION INTEGRAL

It will be observed that in the relay problem we have just discussed, $i(t)$, the resultant current, was written as the sum of the component currents $i_1(t)$, $i_2(t)$ and $i_3(t)$, by the principle of superposition. This principle states that when several electromotive forces are applied simultaneously to a network of fixed parameters each produces its own effect independently of the others. Thus the result of several electromotive forces acting simultaneously is found by adding the effects which would be produced by each acting alone. The principle of superposition was used in Chapter 1, where it was stated that when the indicial admittance function $h(t)$ has been formulated from the set of auxiliary differential equations (2), it is possible to obtain solutions of the canonical equations (1), in the form

$$i_n(t) = \frac{d}{dt}\int_0^t e(t-\psi)h_n(\psi)\,d\psi \quad . \quad . \quad . \quad (3)$$

where $i_n(t)$ represents the current which will flow in the nth mesh of the network in response to an impressed voltage $e(t)$ of arbitrary wave form. Equation (3) can be established in a number of different ways. For example, the current $i(t)$ may be written in operational form as

$$i(t) = e(p)h(p)$$

where $h(p)$ is the operational equivalent of $h(t)$. By virtue of Heaviside's last theorem we may write $h(p)$ as

$$h(p) = \int_0^\infty h(\psi)\varepsilon^{-p\psi}p\,d\psi$$

and consequently

$$e(p)h(p) = p\int_0^\infty \varepsilon^{-p\psi} e(p)h(\psi)\, d\psi$$

Replacing $e(p)$ in the integrand by its equivalent time function and performing the necessary transfer operation, we may write

$$i(t) = \frac{d}{dt}\int_0^t e(t-\psi)h(\psi)\, d\psi$$

which is the same as equation (3). If we make the substitution $x = t - \psi$, and then in order to preserve our original notation, we change the variable of integration from x to ψ, we may write equation (3) in the form

$$i(t) = \frac{d}{dt}\int_0^t h(t-\psi)e(\psi)\, d\psi$$

It will be noted that this formula contains a variable parameter under the sign of integration, and in order to differentiate this expression between the variable limits t and o, we put

$$\phi(t:\psi) = \int_0^t h(t-\psi)e(\psi)\, d\psi$$

then
$$\frac{d}{dt}\phi(t:\psi) = \int_0^t \frac{\partial}{\partial t}h(t-\psi)e(\psi)\, d\psi + h(\mathrm{o})e(t)$$

Thus
$$\frac{d}{dt}\int_0^t h(t-\psi)e(\psi)\, d\psi = h(o)e(t) + \int_0^t h'(t-\psi)e(\psi)\, d\psi$$

Likewise we have

$$\frac{d}{dt}\int_0^t e(t-\psi)h(\psi)\, d\psi = e(\mathrm{o})h(t) + \int_0^t e'(t-\psi)h(\psi)\, d\psi$$

where the primes denote differentiation with respect to the argument. This superposition formula enables us to find the response of a network to an applied electromotive force which varies in any manner whatever, provided we know the value of $h(t)$. The first term on the right-hand side is the current term which results when the switch is closed, and is zero if the voltage passes through the origin at time $t = 0$. The effect of the remainder can be added to this by finding the effect of a step function which closely fits the function $e(t)$ and then passing to the limit. If the superposition formula in any particular case is difficult to evaluate formally, then it is always possible to evaluate it by graphical or mechanical processes. On this fact rests the practical as distinguished from the purely theoretical value of the formula.

THE FOURIER INTEGRAL

If it is found that the steady-state solution of an electric circuit problem is sufficient, then the need for the use of the Heaviside operational calculus discussed in this book does not arise. In cases such as this the well-known symbolic method is sufficient. Perhaps it is now generally recognized that it is not possible to predict completely the performance of an electric network from a single steady-state solution. It is necessary to integrate the individual steady-state responses over an infinite frequency range. Now it is important for the student to realize that there is no fundamental mathematical difference between the Heaviside 'operational' method of solution and the Fourier 'steady-state' summation method. Both methods are ways of solving the canonical equations of electric circuit theory discussed in Chapter 1. It can be shown that the Heaviside Calculus is a simple practical method of evaluating the infinite Fourier integrals which arise in electric circuit problems.

The Fourier solution of the canonical equations discussed in Chapter 1 may be written

$$i(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(\psi)\, d\psi \int_{-\infty}^{\infty} \frac{e^{j\omega(t-\psi)}}{Z(j\omega)} . d\omega \quad . \quad . \quad (23)$$

This infinite integral formulates the current response $i(t)$ of the network, specified by the impedance function $Z(j\omega)$, when an electromotive force $E(t)$ is applied at time $t = 0$. The general method of evaluating this Fourier integral is by means of contour integration in the complex plane and the calculus of residues. By this process it may be applied to the solution of electric circuit problems ; but, generally speaking, this method of solution is very difficult to apply. Compared with the Heaviside method, the Fourier integral solution (23), has no advantages from the standpoint of rigour, and presents formidable mathematical difficulties (even to professional mathematicians) when direct evaluation is attempted.

It is important to note that the Carson integral theorem may be derived from the study of electrical networks in terms of the Fourier integral, and is, in fact, a general relationship resulting from a simple application of the theory of Fourier transforms. In this theory, if

$$f(t) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} e^{\omega t}\phi(\omega)\, d\omega$$

then $$\phi(\omega) = \int_{-\infty}^{\infty} e^{-\omega\lambda} f(\lambda)\, d\lambda$$

is called the Fourier transform of $f(t)$. Now suppose the

indicial admittance function $h(t)$ be written as the Fourier integral

$$h(t) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} e^{\omega t} \cdot \frac{f(\omega)}{\omega} \cdot d\omega$$

then the Fourier transform of this gives the Carson integral relationship

$$f(\omega) = \omega \int_0^\infty e^{-\omega\lambda} h(\lambda)\, d\lambda$$

since $h(t) = 0$ when $t < 0$.

Thus it follows that the operational methods associated with the Carson integral theorem are the working tools of the classical Fourier analysis.

Although the direct evaluation of the Fourier integrals which arise in the analysis of electric circuit problems presents formidable mathematical difficulties, it must be noted that it is possible to use the Fourier integral as a means for obtaining practical design criteria without getting involved in the mathematical pitfalls associated with direct evaluation. For example, in an electrical communication system, the network must be so designed that the received signal is a more or less faithful copy of the sent signal. The departure from this requirement is a measure of the distortion introduced by the network. The Fourier integral provides a simple method of isolating the factors on which this distortion depends and of formulating design criteria for distortionless transmission through the network. Thus it can be shown without the explicit evaluation of the Fourier integrals concerned, the conditions for distortionless transmission through the network are that over the essential range of frequencies contained in the sent signal, the transfer impedance of the network be equalized both as regards amplitude and phase; that is, the amplitude

must be constant and the phase angle linear with the frequency. For another example it is interesting to consider the formulation of the indicial admittance function $h(t)$ as a Fourier integral. From a study of this integral it can be shown that the behaviour of a network under all circumstances is completely determined if either the real or imaginary component of the admittance function $1/Z(j\omega)$ is specified over the entire frequency range. The direct evaluation of $h(t)$ from the Fourier integral formulation is a very complicated procedure, and unless the infinite integral can be recognized, the only practical method of evaluating $h(t)$ is by means of the operational calculus discussed in this book.

In Chapter 5 Carson's integral theorem was obtained as a corollary to Heaviside's last theorem. It is a simple matter to show that by virtue of these theorems the rules and formulae of the operational calculus may be simply but rigorously deduced. Heaviside, however, did not use these theorems to establish operational formulae and processes; but employed them to evaluate the infinite integrals which arise in electromagnetic problems. The theorems are of use either way. Since there are other valid ways of establishing operational formulae, the use that Heaviside made of his last theorem is probably the most valuable. It is sometimes an exceedingly difficult job to evaluate an infinite integral without the aid of this theorem. In the third volume of his *Electromagnetic Theory* Heaviside gives some seventy examples of the evaluation of integrals by the application of the impulse theory discussed in Chapter 5. In these examples he discusses not only the evaluation of ordinary infinite integrals but shows how the fundamental Fourier theorems may be readily established by the application of his impulse ideas. He also deals with Bessel theorems, elliptic functions and, among other things, with convergent and divergent series. In all cases his operational treatments are interpreted in accordance with the impulse ideas discussed in Chapter 5. As a simple example,

suppose that the infinite integral $\int_0^\infty J_0(2\sqrt{xt})\,dx$ is encountered. Replacing the Bessel function $J_0(2\sqrt{xt})$ by its operational equivalent $e^{-\frac{x}{p}}[1]$, and then integrating with respect to x, we obtain $p[1]$. The value of the infinite integral is therefore zero, except for the special case where $t = 0$, in which case it is infinite. This result, which has been obtained so easily, would be very difficult to establish by other methods. Only this one example will be reviewed here, for our subject is Heaviside's electric circuit theory and not the evaluation of infinite integrals by special methods invented by Heaviside. A complete discussion of the evaluation of infinite integrals by means of Heaviside's impulse functions is beyond the scope of an elementary book such as this.

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